Notes on Game Theory

md.mottakin.chowdhury

October 2018

1 Fibonacci Nim

Rules: Fibonacci nim is played by two players, who alternate removing coins or other counters from a pile of coins. On the first move, a player is not allowed to take all of the coins, and on each subsequent move, the number of coins removed can be any number that is at most twice the previous move. According to the normal play convention, the player who takes the last coin wins. Or according to the Misère game, the player who takes the last coin loses.

This game should be distinguished from a different game, also called Fibonacci nim, in which players may remove any Fibonacci number of coins on each move.

Strategy: The optimal strategy in Fibonacci nim can be described in terms of the "quota" q (the maximum number of coins that can currently be removed: all but one on the first move, and up to twice the previous move after that) and the Zeckendorf representation of the current number of coins as a sum of non-consecutive Fibonacci numbers. A given position is a losing position (for the player who is about to move) when q is less than the smallest Fibonacci number in this representation, and a winning position otherwise. In a winning position, it is always a winning move to remove all the coins (if this is allowed) or otherwise to remove a number of coins equal to the smallest Fibonacci number in the Zeckendorf representation. When this is possible, the opposing player will necessarily be faced with a losing position, because the new quota will be smaller than the smallest Fibonacci number in the Zeckendorf representation of the remaining number of coins. From a losing position, any move will lead to a winning position.[1]

In particular, when there is a Fibonacci number of coins in the starting pile, the position is losing for the first player (and winning for the second player). However, when the starting number of coins is not a Fibonacci number, the first player can always win with optimal play.[2]

For the Misere game of this game, if there are initially n coins, then the first player can remove n-1 coins and leave 1 coin to win.

Example: For example, suppose that there are initially 10 coins. The Zeckondorf representation is $10 = 8 + 2$, so a winning move by the first player would be to remove the smallest Fibonacci number in this representation, 2, leaving 8 coins. The second player can remove at most 4 coins, but removing 3 or more would allow the first player to win immediately, so suppose that the second player takes 2 coins. This leaves $6 = 5 + 1$ coins, and the first player again takes the smallest Fibonacci number in this representation, 1, leaving 5 coins. The second player could take two coins, but that would again lose immediately, so the second player takes only one coin, leaving $4 = 3 + 1$. The first player again takes the smallest Fibonacci number in this representation, 1, leaving 3 coins. Now, regardless of whether the second player takes one or two coins, the first player will win the game in the next move.

2 Misere Nim

Here is Bouton's method for playing misère nim optimally. Play it as you would play nim under the normal play rule as long as there are at least two heaps of size greater than one. When your opponent finally moves so that there is exactly one pile of size greater than one, reduce that pile to zero or one, whichever leaves an odd number of piles of size one remaining.

This works because your optimal play in nim never requires you to leave exactly one pile of size greater than one (the nim sum must be zero), and your opponent cannot move from two piles of size greater than one to no piles greater than one. So eventually the game drops into a position with exactly one pile greater than one and it must be your turn to move.

3 Turning Turtles

A horizontal line of n coins is laid out randomly with some coins showing heads and some tails. A move consists of turning over one of the coins from heads to tails, and in addition, if desired, turning over one other coin to the left of it (from heads to tails or tails to heads).

This game is just nim in disguise if an H in place n is taken to represent a nim pile of n chips.

4 Staircase Nim

In Staircase Nim, there is a staircase with n steps, indexed from 0 to $n-1$. In each step, there are zero or more coins. Two players play in turns. In his/her move a player can choose a step $i > 0$ and move one or more coins to step $i-1$. The player who is unable to make a move lose the game. That means the game ends when all the coins are in step 0.

We can divide the steps into two types, odd steps, and even steps. Now let's think what will happen if a player A move a coin from an even step to an odd step. Player B can move those coins to an odd position and the state of the game won't change.

But if A move a coin from an odd step to an even step, similar logic won't work. Because there can be situation where player B won't be able to move those coins to another odd step to restore the state.

From this we can agree that coins in even steps are useless, they don't affect game state. If I am in a winning position and you move a coin from an even step, I will move those coins again to another even step and will remain in a winning position.

Now we agreed that only coins to odd steps count. If you take one or more coins from an odd step and move them to an even step, the coins become useless! Remember even steps are useless, So moving to even step is just like throwing them away. Now we can imagine coins in an odd-step as a pile of stones in a standard Nim game.

Now its easy, just find the xorsum of all odd steps and we are done!

5 Whythoff's Game

Wythoff's game is a two-player mathematical game of strategy, played with two piles of counters. Players take turns removing counters from one or both piles; when removing counters from both piles, the numbers of counters removed from each pile must be equal. The game ends when one person removes the last counter or counters, thus winning.

An equivalent description of the game is that a single chess queen is placed somewhere on a large grid of squares, and each player can move the queen towards the lower left corner of the grid: south, west, or southwest, any number of steps. The winner is the player who moves the queen into the corner.

Any position in the game can be described by a pair of integers (n, m) with $n \geq m$, describing the size of both piles in the position or the coordinates of the queen. The strategy of the game revolves around cold positions and hot positions: in a cold position, the player whose turn it is to move will lose with best play, while in a hot position, the player whose turn it is to move will win with best play. The optimal strategy from a hot position is to move to any reachable cold position.

The classification of positions into hot and cold can be carried out recursively with the following three rules:

- $(0, 0)$ is a cold position.
- Any position from which a cold position can be reached in a single move is a hot position.
- If every move leads to a hot position, then a position is cold.

For instance, all positions of the form $(0, m)$ and (m, m) with $m > 0$ are hot, by rule 2. However, the position $(1, 2)$ is cold, because the only positions that can be reached from it, $(0, 1)$, $(0, 2)$, and $(1, 1)$, are all hot. The cold positions (n, m) with the smallest values of n and m are $(0, 0), (1, 2), (3, 5), (4, 7), (6, 10)$ and $(8, 13)$.

Wythoff discovered that the cold positions follow a regular pattern determined by the golden ratio. Specifically, if k is any natural number and

$$n_k = floor(k\phi) = m_k\phi - m_k$$
$$m_k = floor(k\phi^2) = ceil(n_k\phi) = n_k + k$$

where ϕ is the golden ratio and we are using the floor function, then (n_k, m_k) is the k^{th} cold position.

6 Green Hackenbush

The game of Hackenbush is played by hacking away edges from a rooted graph and removing those pieces of the graph that are no longer connected to the ground. A rooted graph is an undirected graph with every edge attached by some path to a special vertex called the root or the ground. The ground is denoted in the figures that follow by a dotted line.

Green Hackenbush on Trees: Played with bamboo stalks, Green Hackenbush is just nim in a rather transparent disguise. But what happens if we allow more general structures than these simple bamboo stalks? Suppose we have the "forest" of three rooted trees.

Again a move consists of hacking away any segment and removing that segment and anything not connected to the ground. Since the game is impartial, the general theory of Section 4 tells us that each such tree is equivalent to some nim pile, or if you will, to some bamboo stalk. The problem is to find the Sprague-Grundy values of each of the trees.

This may be done using the following principle, known in its more general form as the **Colon Principle**. When branches come together at a vertex, one may replace the branches by a non-branching stalk of length equal to their nim sum.

Green Hackenbush on General Rooted Graphs: We now consider arbitrary graphs. These graphs may have circuits and loops and several segments may be attached to the ground. **The Fusion Principle** states that the vertices on any circuit may be fused without changing the Sprague-Grundy value of the graph. We see more generally that a circuit with an odd number of edges reduces to a single edge, and a circuit with an even number of edges reduces to a single vertex.