Math Refresher for DS

Lecture 1

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ai

Linear Algebra: Core Objects

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Example: $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \in \mathbb{R}^5$

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- $A \in \mathbb{R}^{m \times n}$ - a matrix with m rows and n columns

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2\times 3}$

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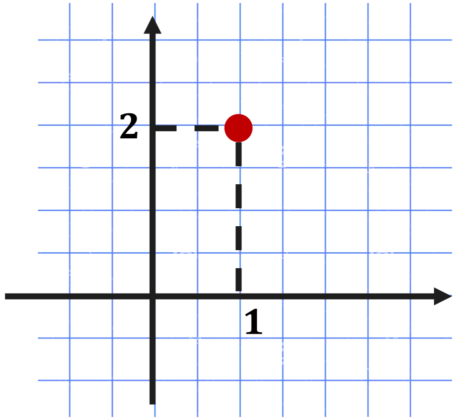

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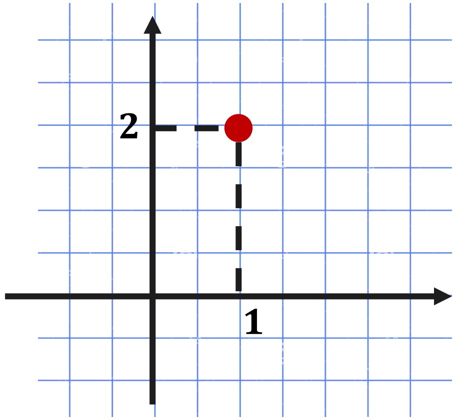
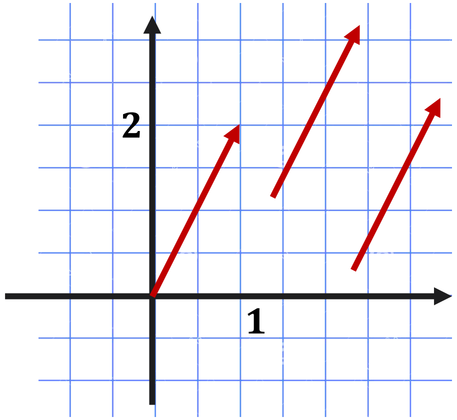


- Direction + length



Vector Spaces

Vector Space: Definition

- A real-valued **vector space** $(V, +, \ \cdot)$ is a set of vectors V with two operations

$$(1)\ +: \ V \times V \to V, \qquad (2)\ \cdot: \ \mathbb{R} \times V \to V$$

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that satisfy the following properties (axioms):

	Property	Meaning
1.	**Associativity** of addition	$x + (y + z) = (x + y) + z$
2.	**Commutativity** of addition	$x + y = y + x$
3.	**Identity element** of addition	$\exists 0 \in V:\ \forall x \in V \quad 0 + x = x$
4.	**Identity element** of scalar multiplication	$\forall x \in V \quad 1 \cdot x = x$
5.	**Inverse element** of addition	$\forall x \in V\ \exists -x \in V:\ \ x + (-x) = 0$
6.	**Compatibility** of scalar multiplication	$\alpha(\beta x) = (\alpha\beta)x$
7.	**Distributivity**	$(\alpha + \beta)x = \alpha x + \beta x$
8.		$\alpha(x + y) = \alpha x + \alpha y$

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Let’s define vector operations!

Operations with Vectors

1. Sum of two vectors:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n, \qquad x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_2 \end{bmatrix} \in \mathbb{R}^n$$

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2. Multiplying by a scalar:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \qquad \alpha \in \mathbb{R}, \qquad \alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Operations with Vectors: Example

$x, y \in \mathbb{R}^3$:

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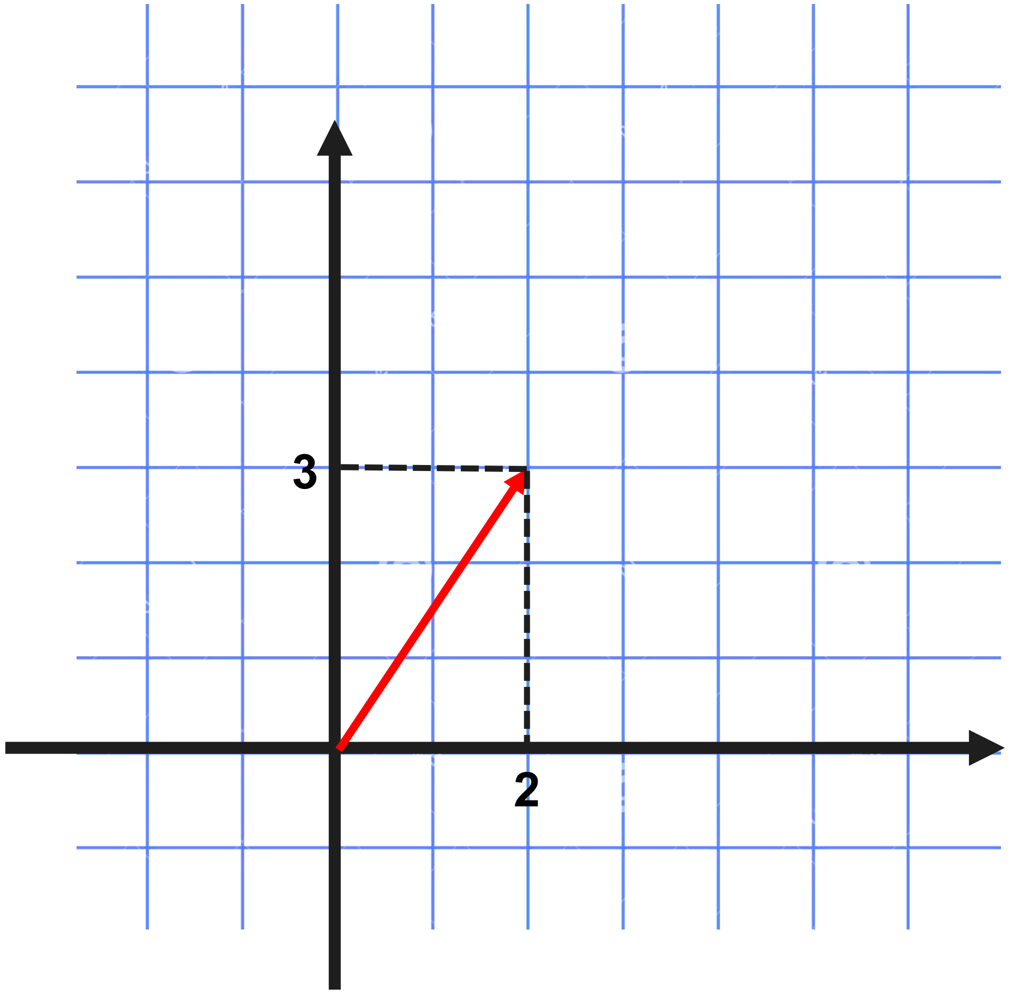
Sum:

$$x + y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

Vector Operations: Geometrical Interpretation

Vectors: Geometrical Interpretation

$\vec{a} = [2, 3]$



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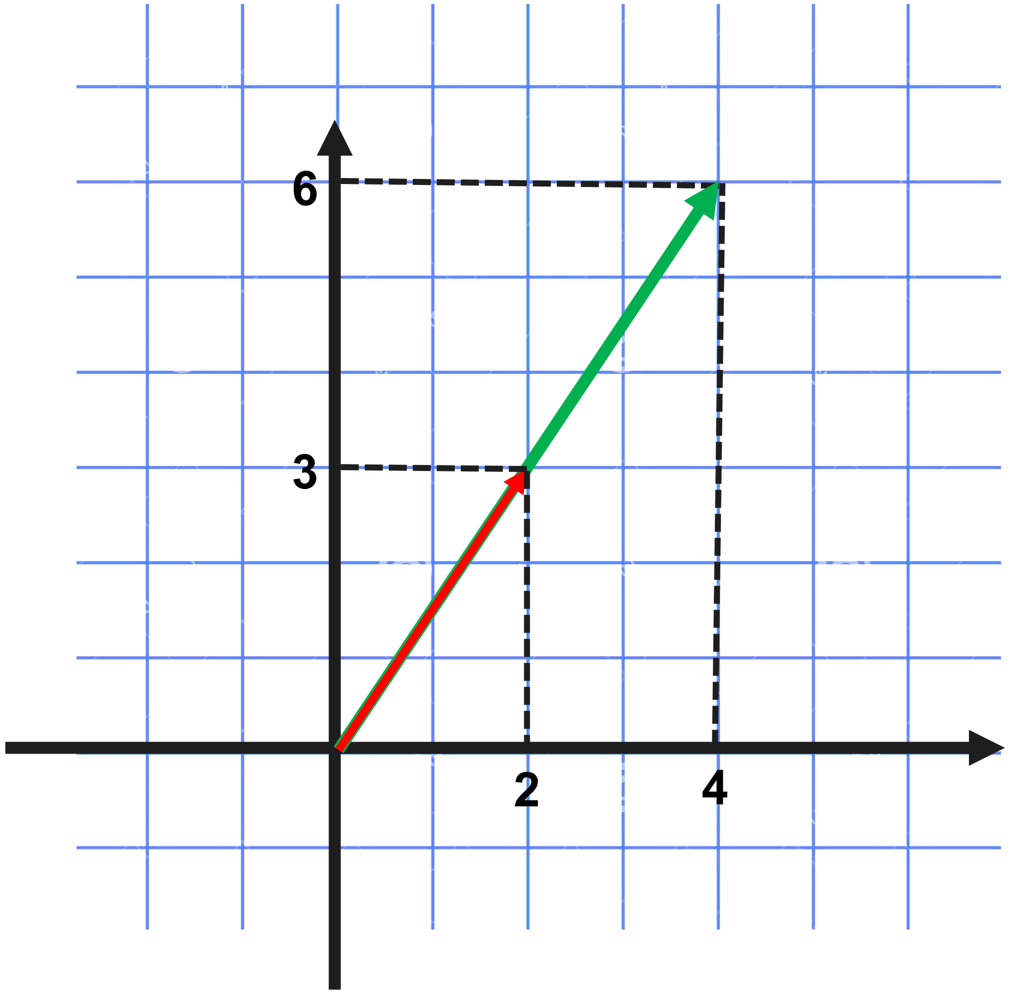

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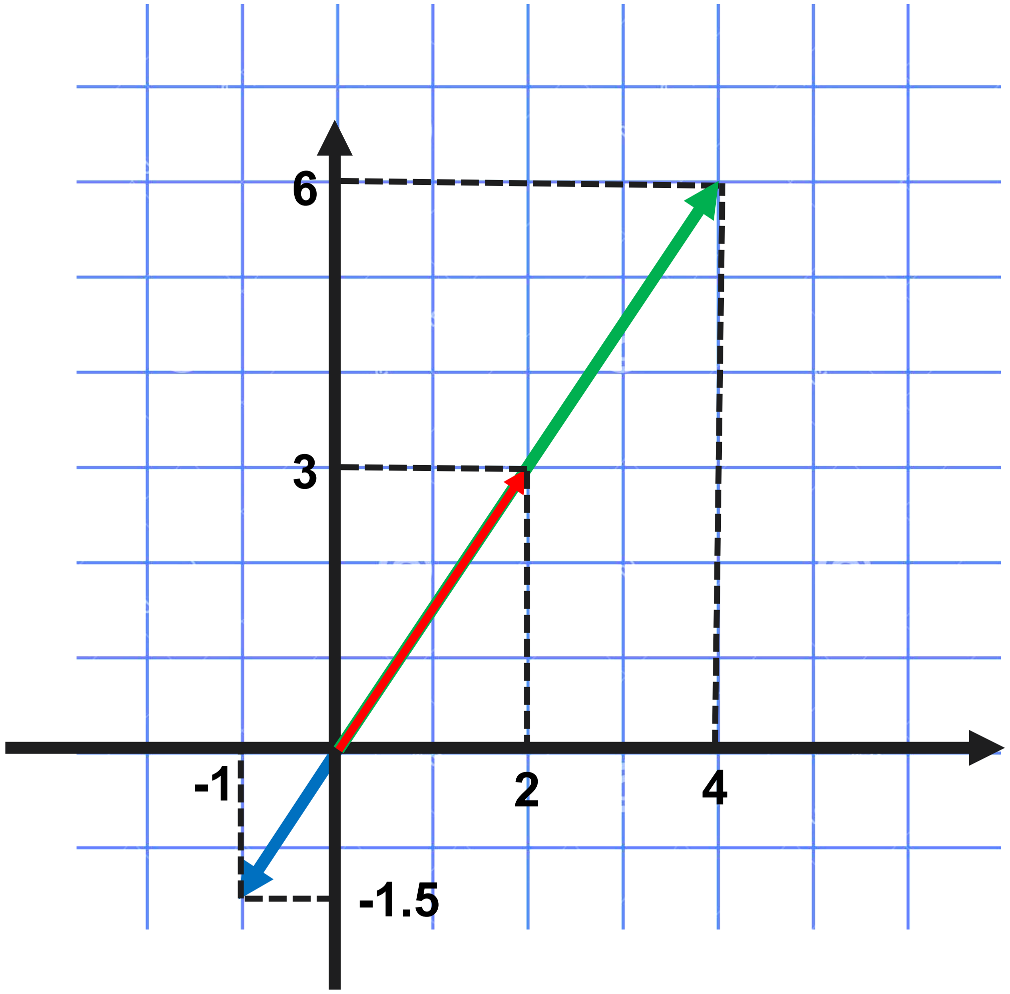

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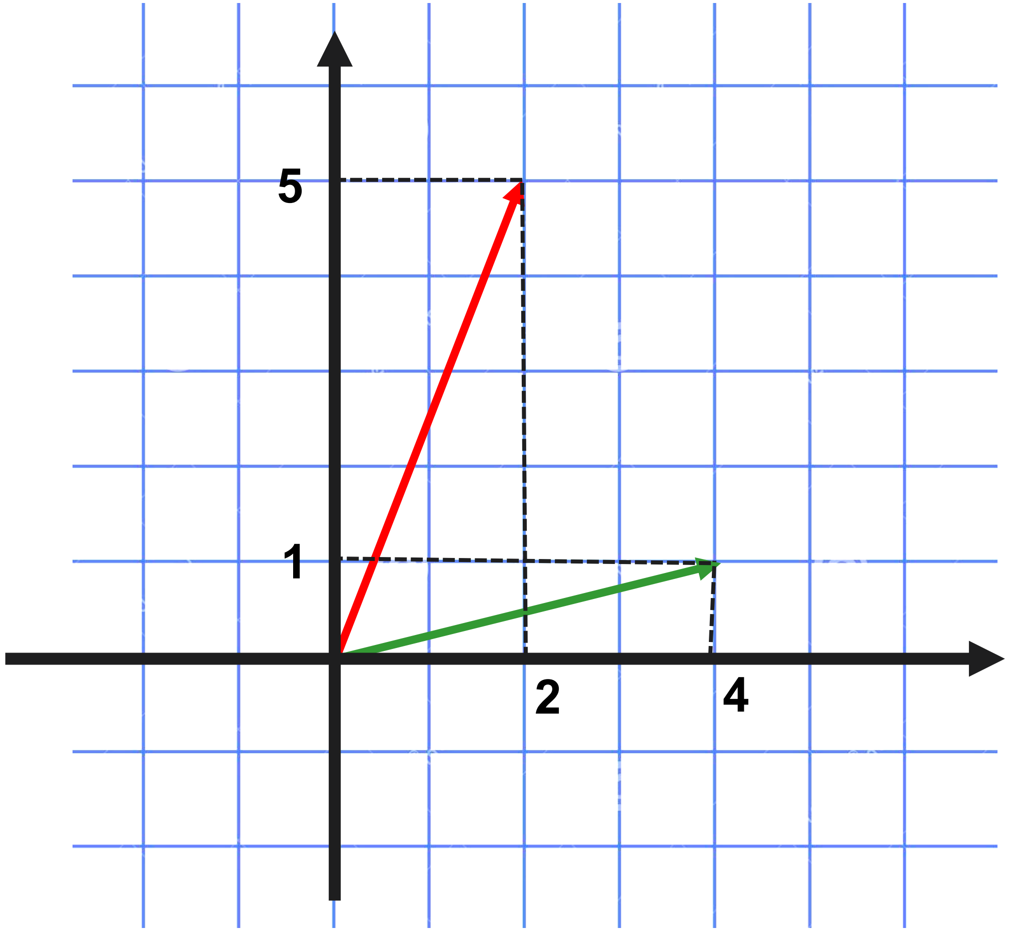

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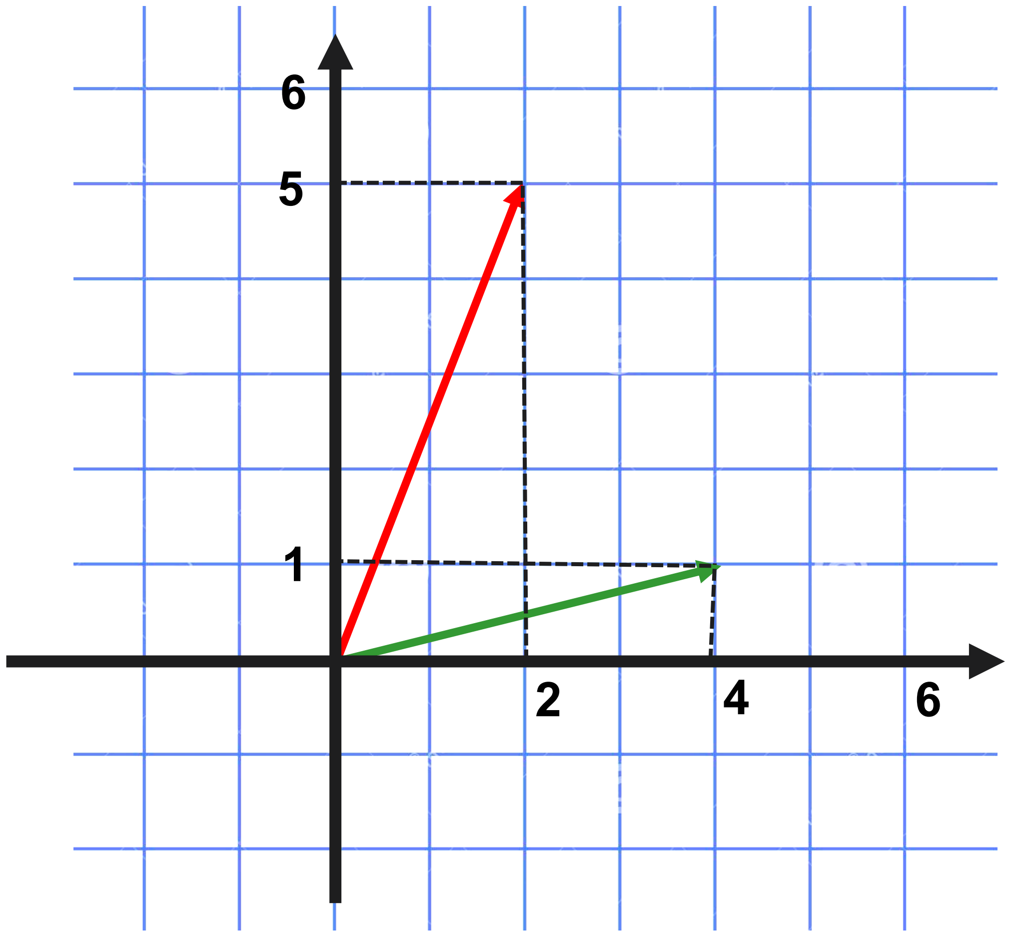

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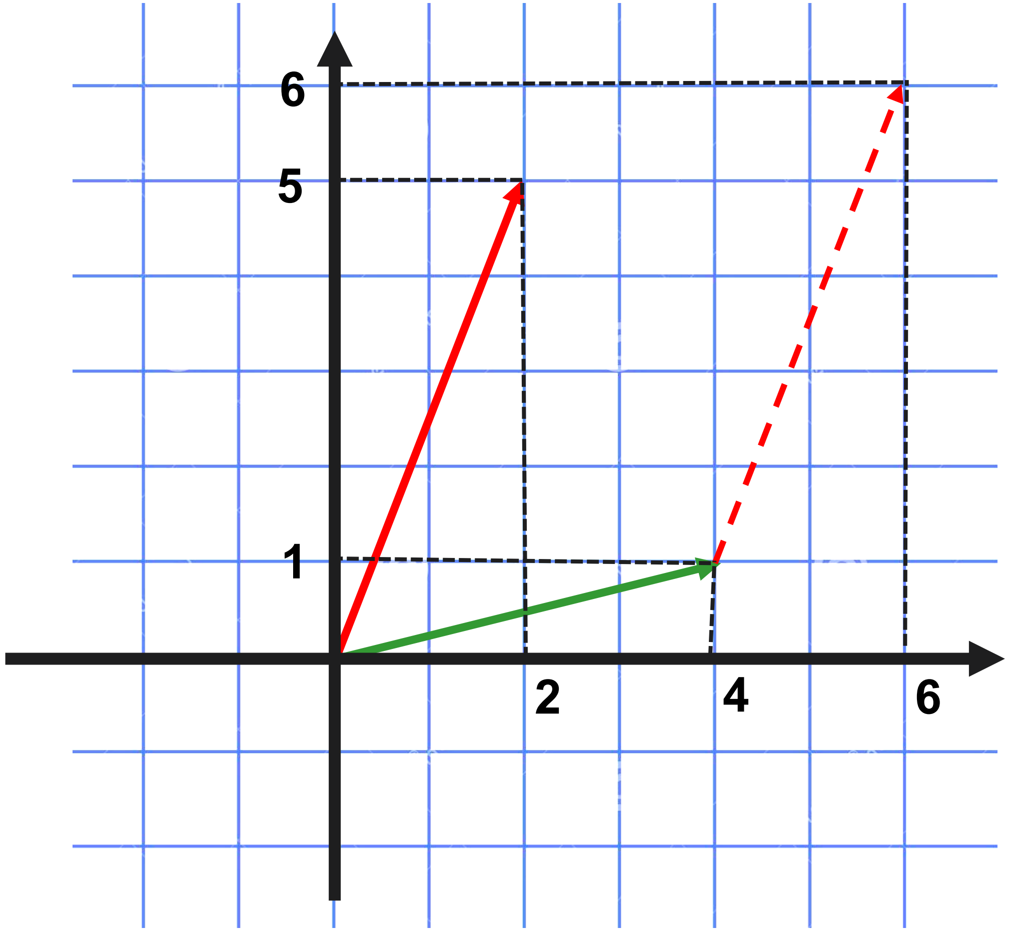

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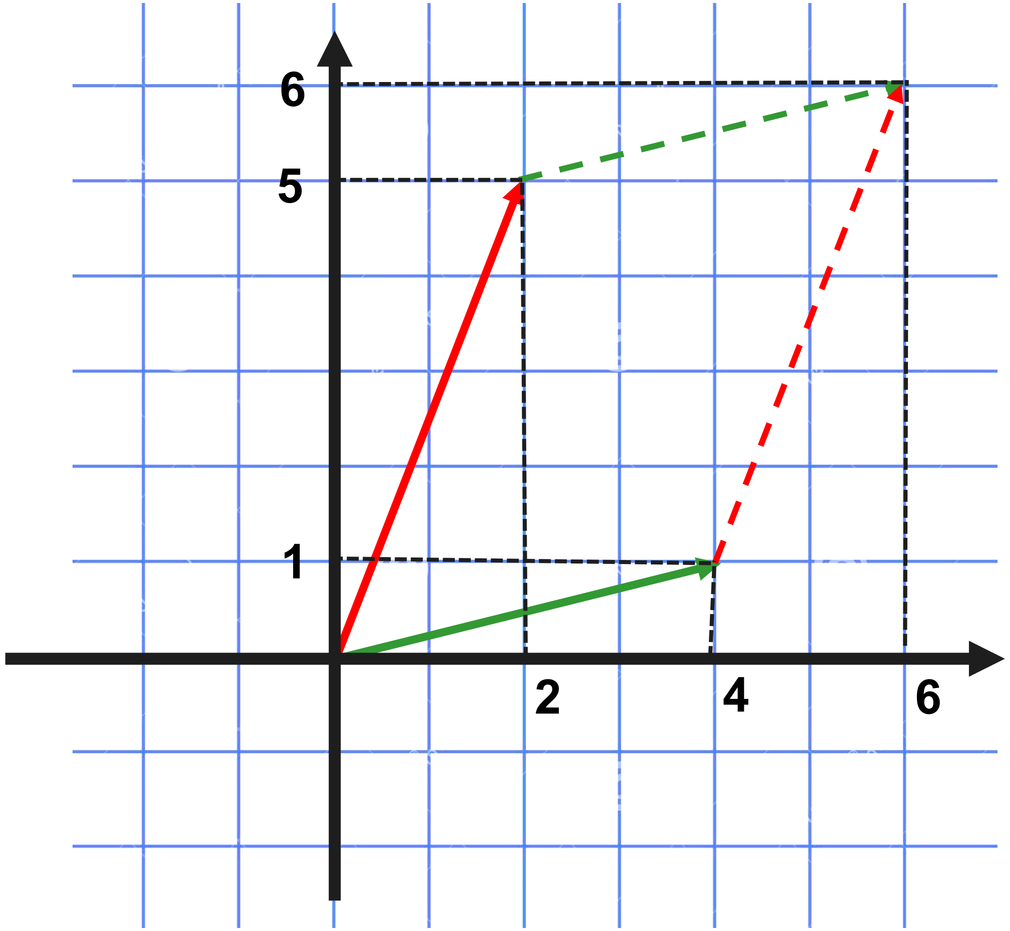

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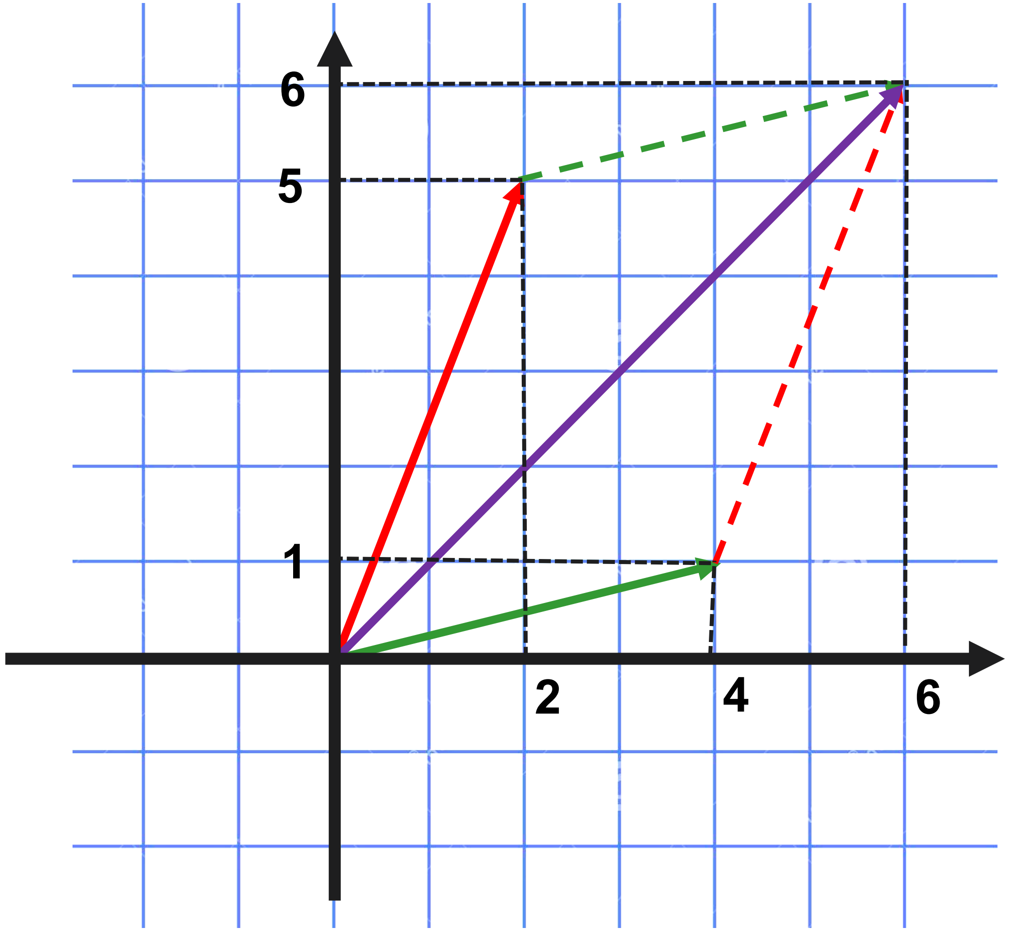

Back to Vector Spaces

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2. Multiplying by a scalar:

satisfy axioms (1) – (8)
(check it yourself)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \alpha \in \mathbb{R}, \qquad \alpha x = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \in \mathbb{R}^n$$

Vector Spaces

$(\mathbb{R}^n, +, \cdot), n \in \mathbb{N}$ - a vector space with operations

1. vector addition:

$$x + y = (x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

2. multiplication by a scalar:

$$\alpha x = \alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$$

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- $\mathbb{P}^n$ - a set of polynomials of degree $\leq n$ with real coefficients
    - *Example:* $\mathbb{P}^3 = \{ax^3 + bx^2 + cx + d \mid a, b, c, d \in \mathbb{R}\}$

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- We can add up polynomials:

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- And we can multiply them by a real number:

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- These operations satisfy axioms (1) – (8)!

$\rightarrow (\mathbb{P}^n, +, \cdot)$ is also a vector space!

Inner Product

Inner Product

- Inner product is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties:
    - *Symmetric:* $\forall x, y \in V \;\; \langle x, y \rangle = \langle y, x \rangle$
    - *Positive definite:* $\forall x \in V \setminus \{0\} \;\; \langle x, x \rangle > 0$ and $\langle x, 0 \rangle = 0$.

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- For $x = [x_1, \dots, x_n],\ y = [y_1, \dots, y_n] \in \mathbb{R}^n$

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- Example:

$$x = [1, 2, 3, 4], \qquad y = [-1, 0, 1, 2]$$

$$(x, y) = 1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 1 + 4 \cdot 2 = -1 + 0 + 3 + 8 = 10$$

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- *Note: there're inner products different from dot product.*

Norms

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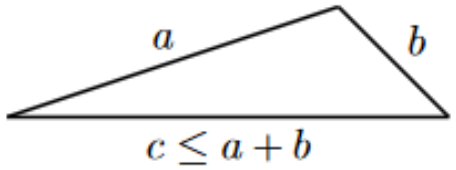
  - *Positive definite:* $\forall x \in \mathbb{V} \quad \|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$
  - *Triangle inequality:* $\forall x, y \in \mathbb{V} \ \ \|x + y\| \leq \|x\| + \|y\|$



Examples of Norms

Manhattan Norm

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- Examples:

$$\|[1,2,3]\|_1 = 1 + 2 + 3 = 6$$

$$\|[1,0]\|_1 = 1 + 0 = 1$$

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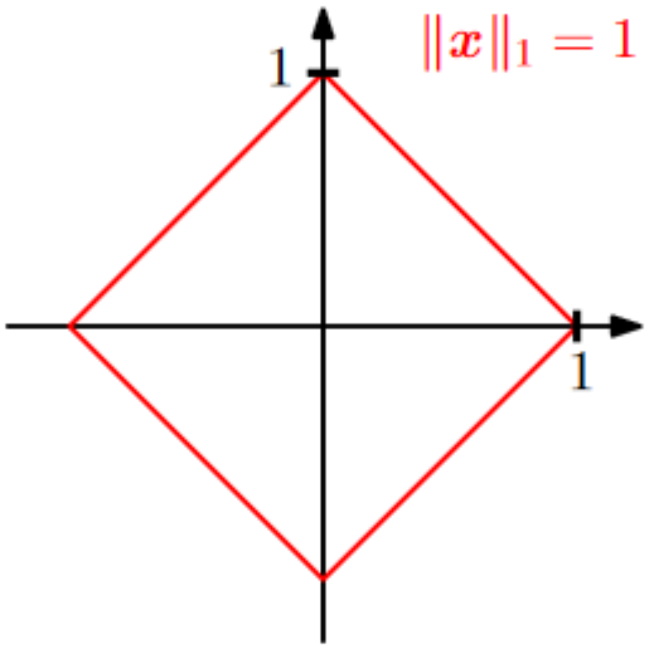

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- Examples:

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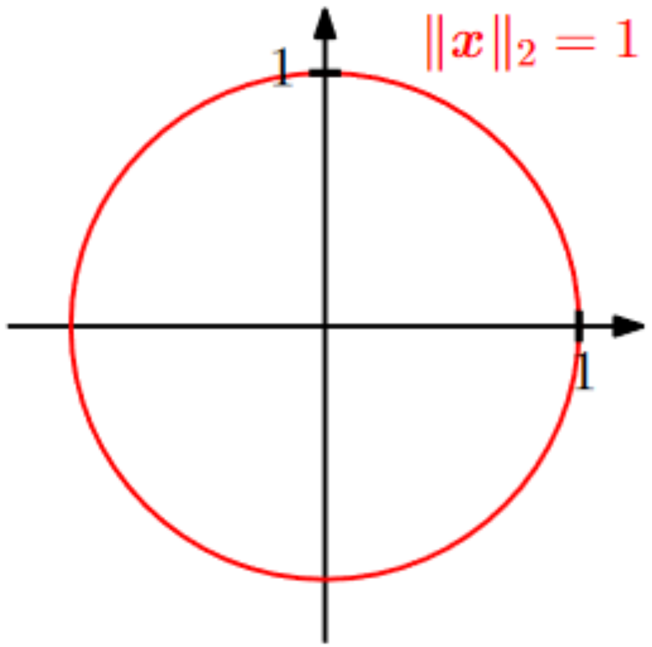

Other norms

- In general, for $x = [x_1, \dots, x_n] \in \mathbb{R}^n$ an $\boldsymbol{\ell_p}$**-norm** is defined as follows:

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  - ℓ_1 - Manhattan norm $\|\cdot\|_1$;
  - ℓ_2 - Euclidian norm $\|\cdot\|$ (default);
  - ℓ_∞: $\| x \|_\infty = \max_i |x_i|$

    Example: $\|[1,2,3]\|_\infty = 3, \quad \|[1,0]\|_\infty = 1, \quad \|[-1,0.5]\|_\infty = 1.$

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- (!) Not every norm is induced by an inner product.
  Example: Manhattan norm.

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Cauchy-Schwarz Inequality

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- For dot product and Euclidian norm:

$$|(x, y)| \leq \|x\|_2 \cdot \|y\|_2$$

Distance between Vectors

- Distance between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined as

$$d(x,y) := \|x-y\| = \sqrt{\langle x-y, x-y \rangle}.$$

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- Distance between two vectors x and y is defined as

$$d(x,y) := \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

- For dot product and Euclidian norm, we get *Euclidian distance*:

$$d(x,y) = \|x - y\|_2 = \sqrt{(x - y, x - y)} =$$

$$= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

Angles and Orthogonality

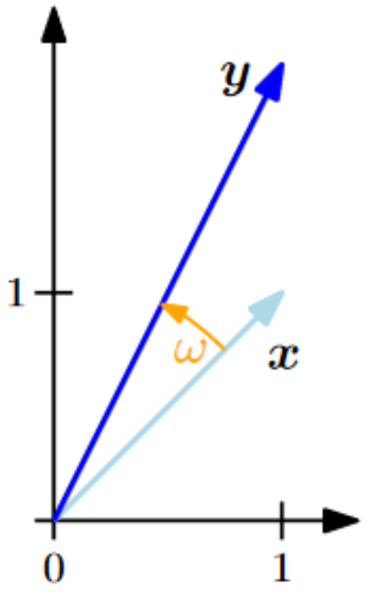
Angle between Two Vectors

- Inner product also captures the geometry of vector space by defining the angle between two vectors.

- Remember Cauchy-Schwarz inequality:

$$|(x, y)| \leq \|x\| \cdot \|y\|$$

$$-1 \leq \frac{(x, y)}{\|x\| \cdot \|y\|} \leq 1$$



$\omega\!: \cos\omega = \frac{(x,y)}{\|x\|\cdot\|y\|}$ - angle between x and y.

Angle between Two Vectors: Example

- What is the angle ω between $x = [5, 0]$ and $y = [1, 1]$?

$$\omega = \arccos \frac{(x, y)}{\|x\|\|y\|} = \arccos \frac{5 \cdot 1 + 0 \cdot 1}{\sqrt{5^2 + 0^2} \cdot \sqrt{1^2 + 1^2}} = \arccos \frac{5}{5\sqrt{2}} = \arccos \frac{\sqrt{2}}{4} = \frac{\pi}{4}.$$

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$$\omega = \arccos \frac{(x, y)}{\|x\|\|y\|} = \arccos \frac{0}{\sqrt{2} \cdot \sqrt{2}} = \arccos 0 = \frac{\pi}{2}.$$

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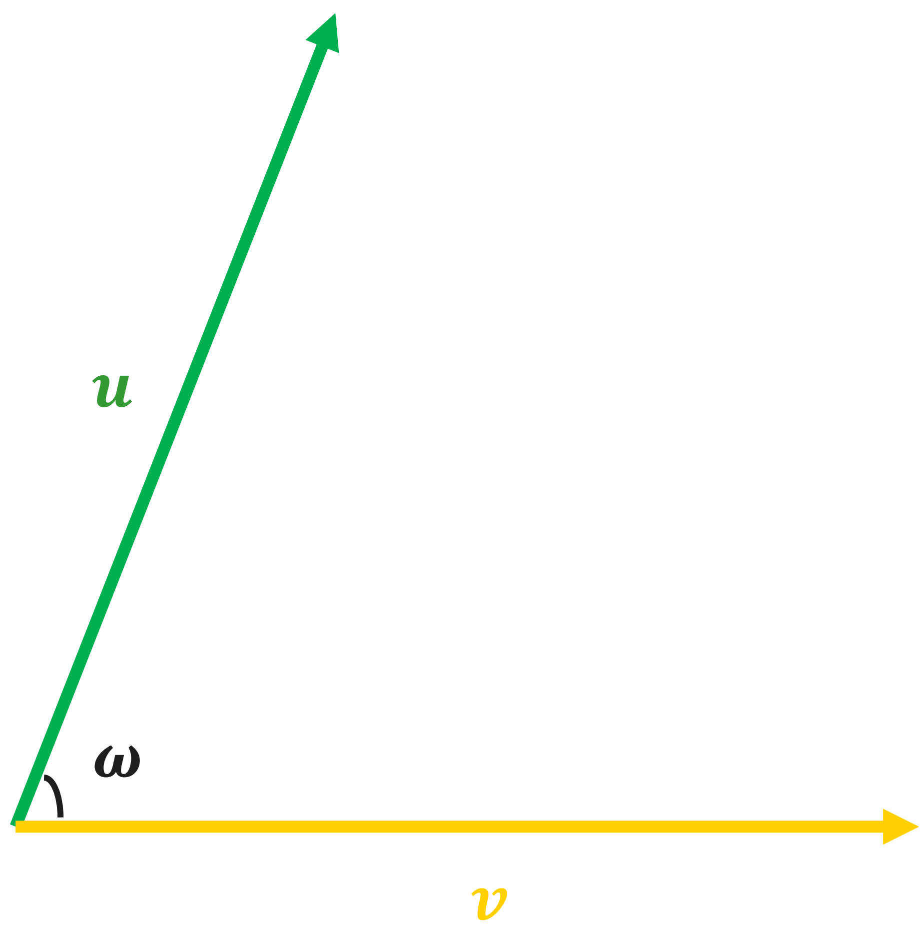
$$x = [1, 0], \qquad y = [0,1], \qquad (x, y) = 0, \qquad \|x\| = \|y\| = 1 \;\rightarrow$$

x and y are *orthonormal*.

Orthogonal Projection

- Suppose we have two vectors u and v.



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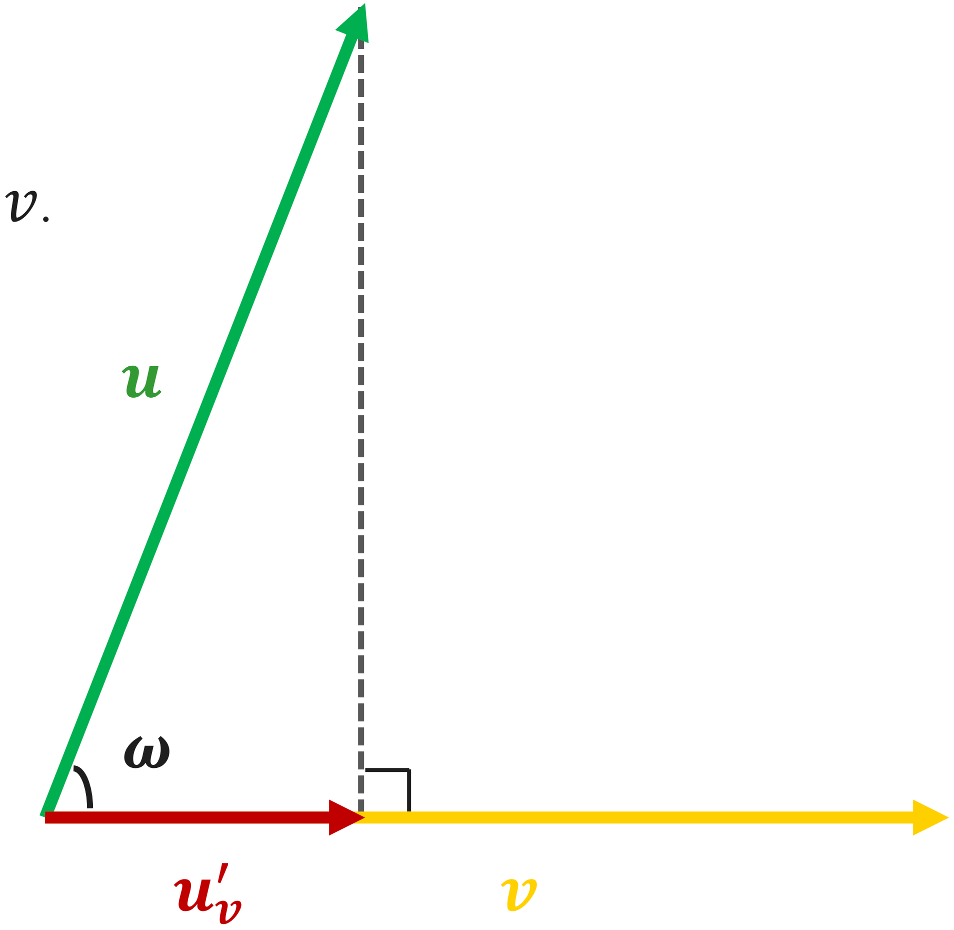

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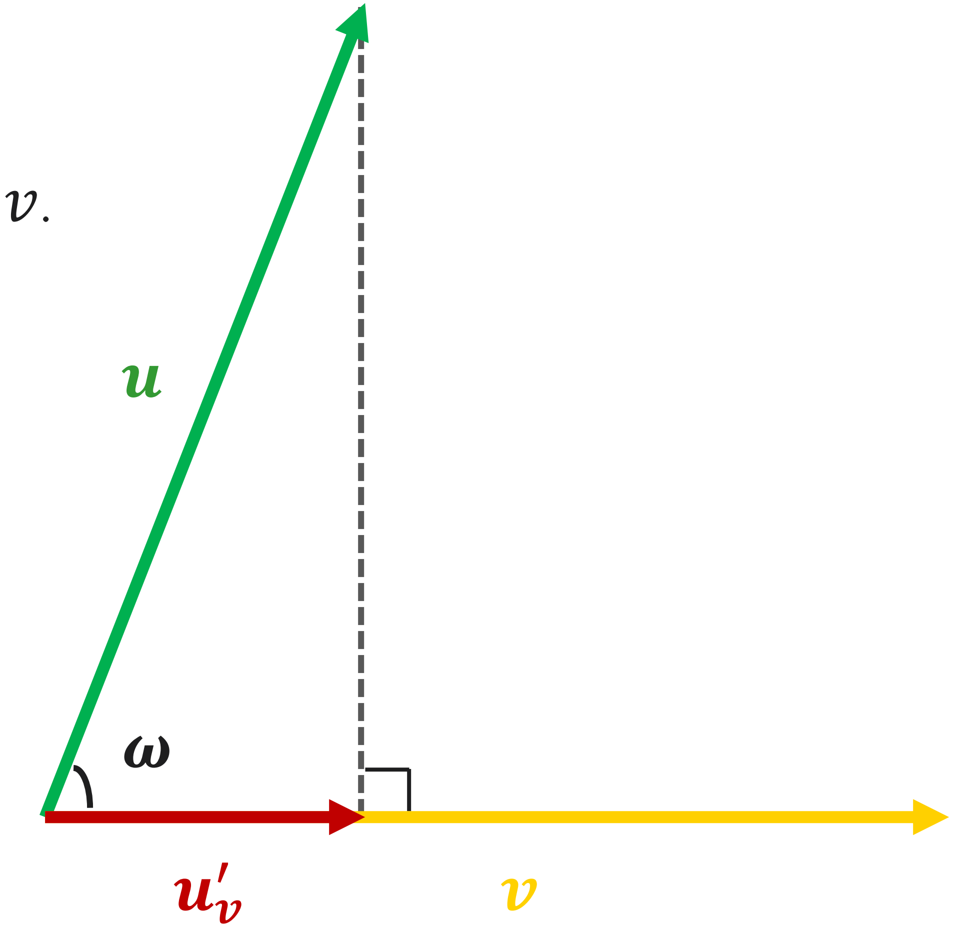

Orthogonal Projection

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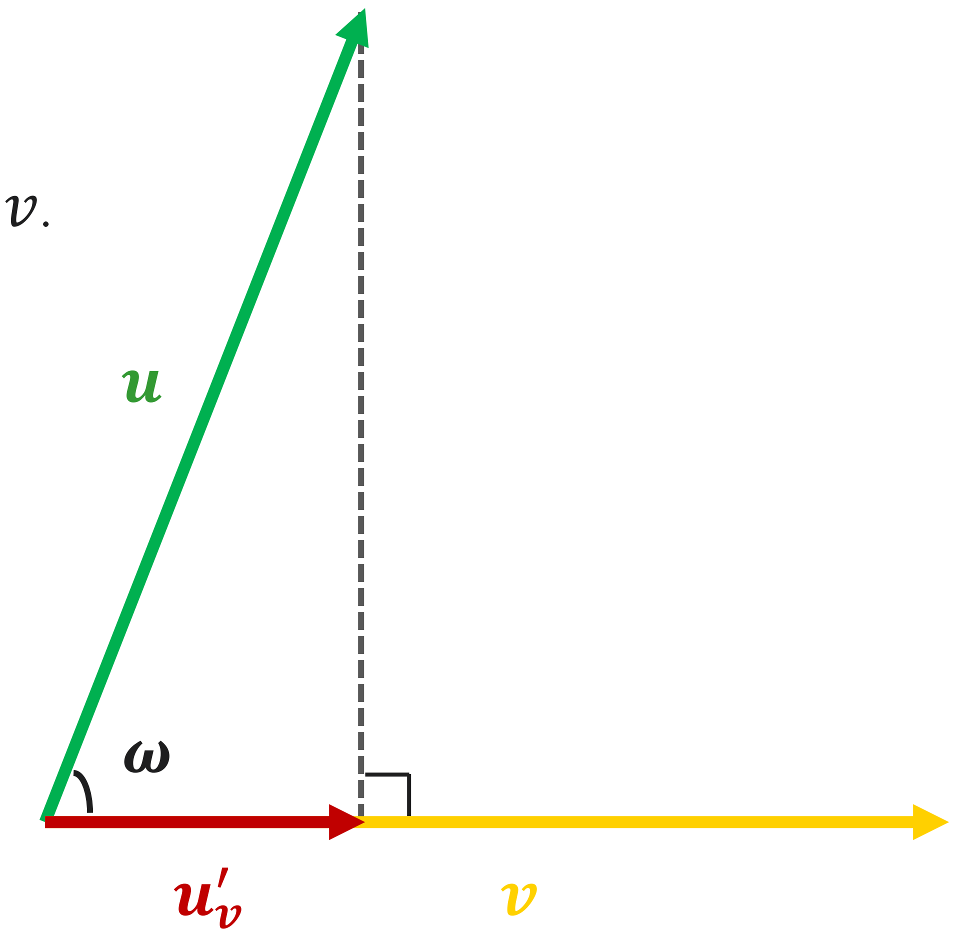
If $\mathbf{0} \le \boldsymbol{\omega} \le \mathbf{90}$

$$(u, v) = \|u\|\|v\| \cos \boldsymbol{\omega} = \|u\|\|v\| \frac{\|\boldsymbol{u'_v}\|}{\|\boldsymbol{u}\|} =$$
$$= \|u'_v\|\|v\|$$



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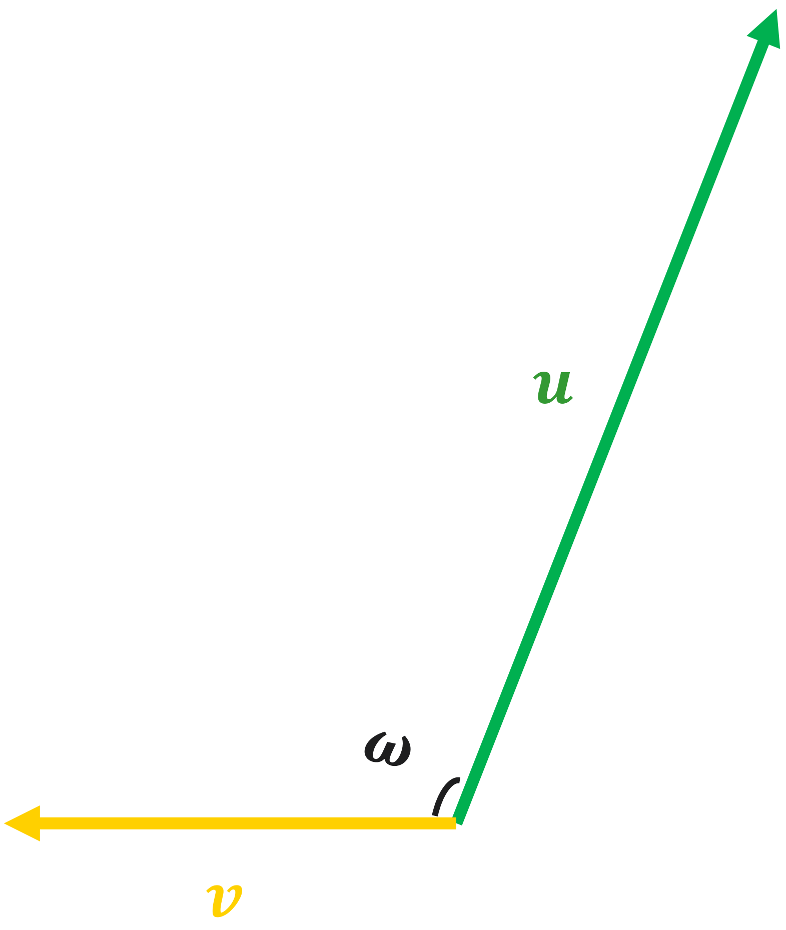

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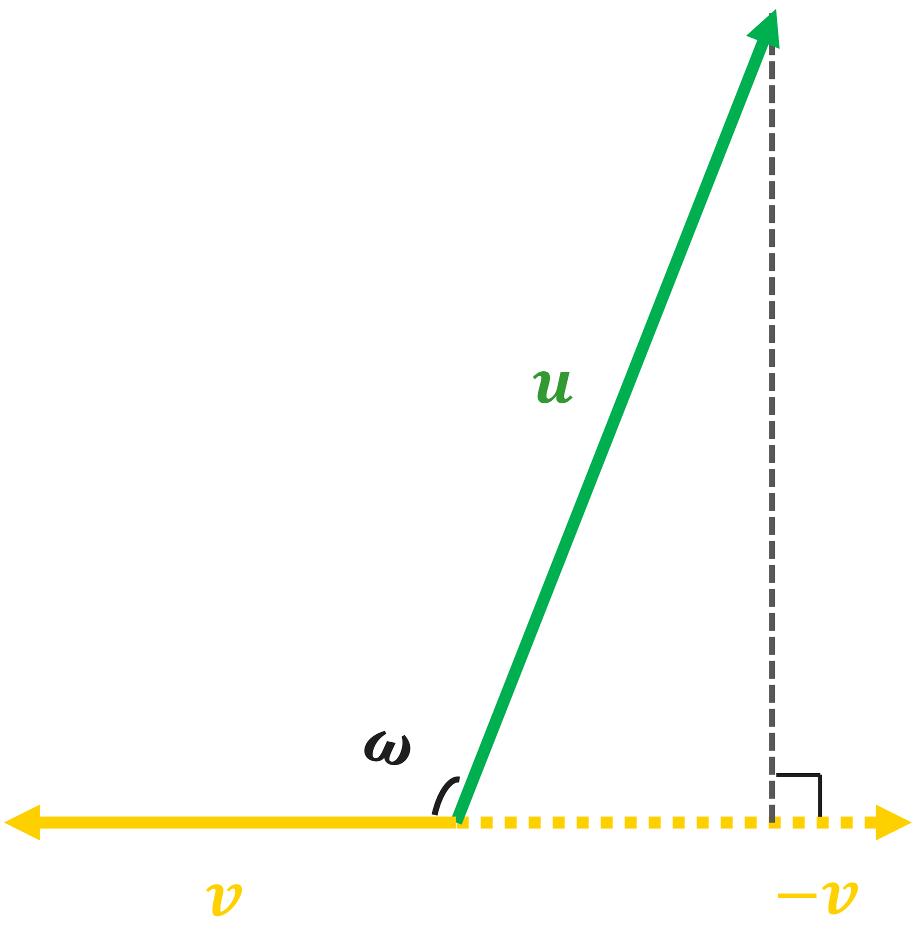

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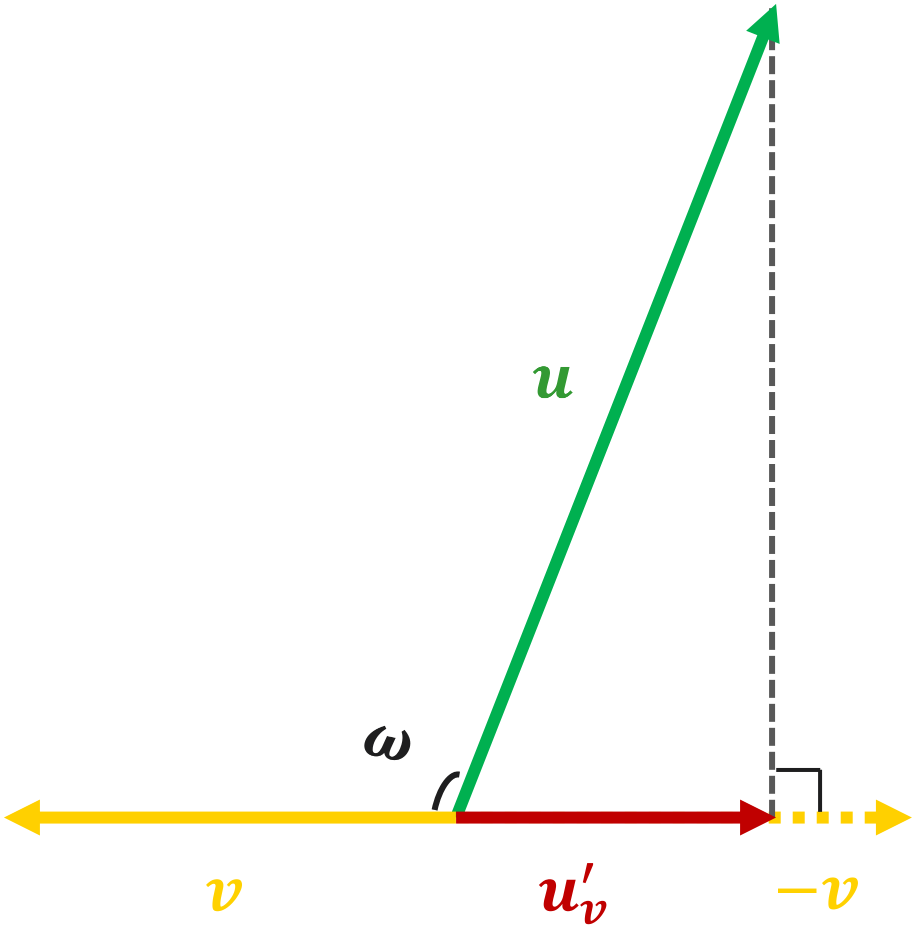

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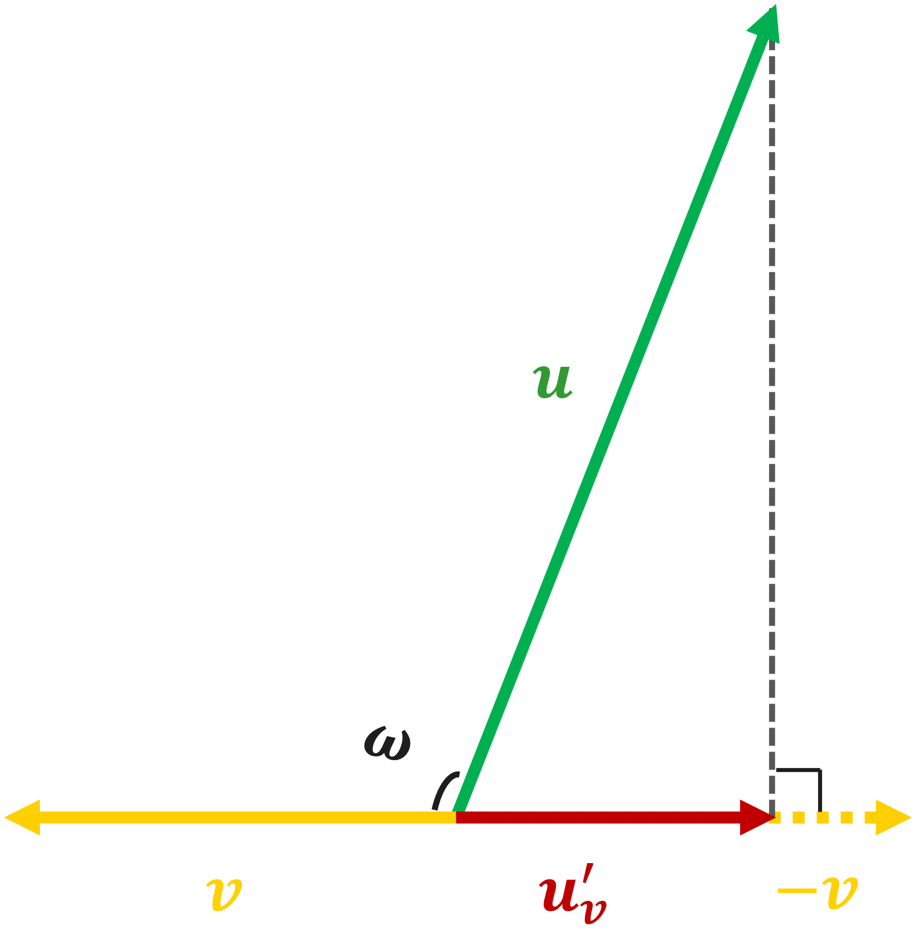

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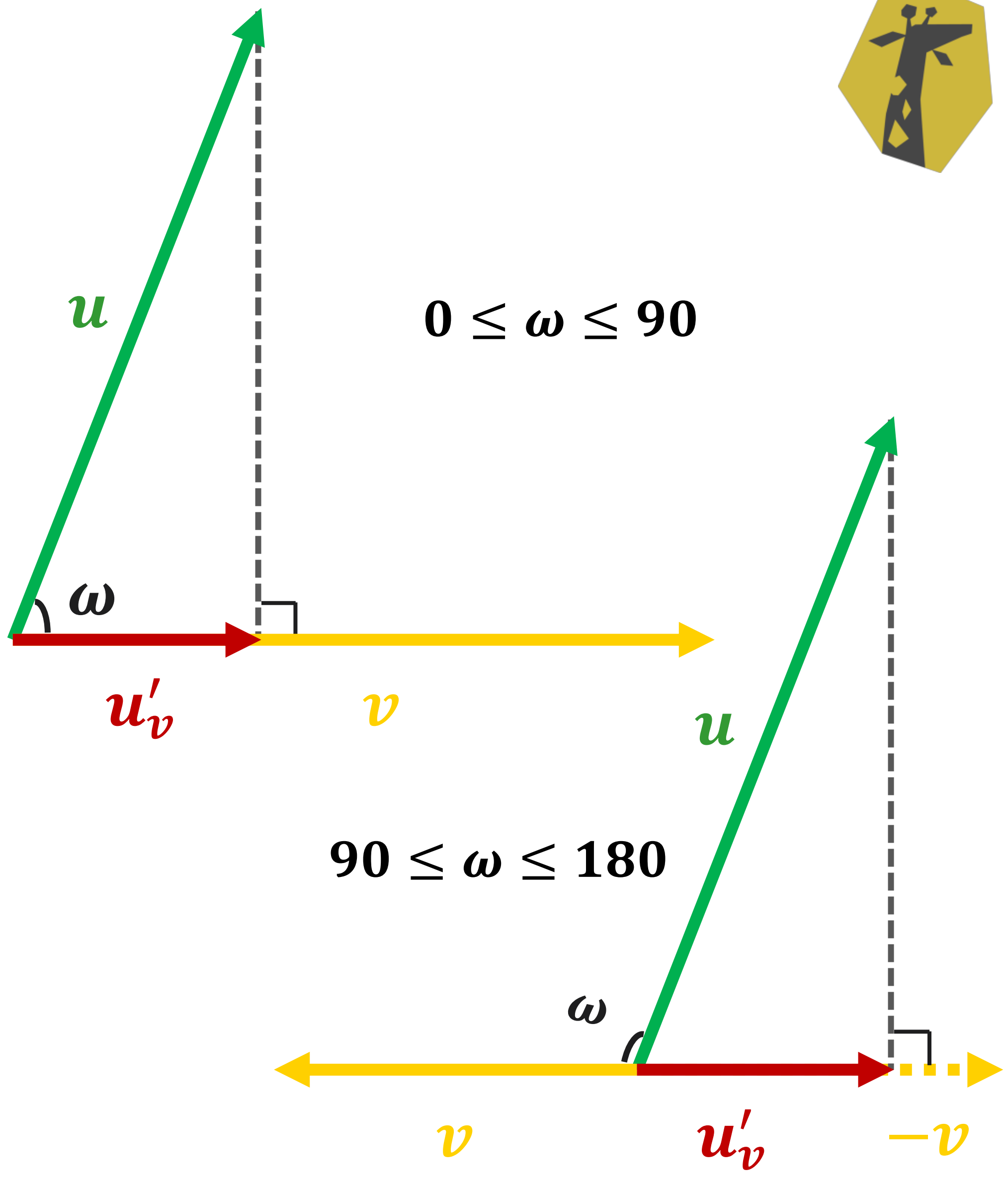
$$|(u, v)| = \|u'_v\|\|v\| \leftrightarrow \|u'_v\| = \frac{|(u, v)|}{\|v\|}$$



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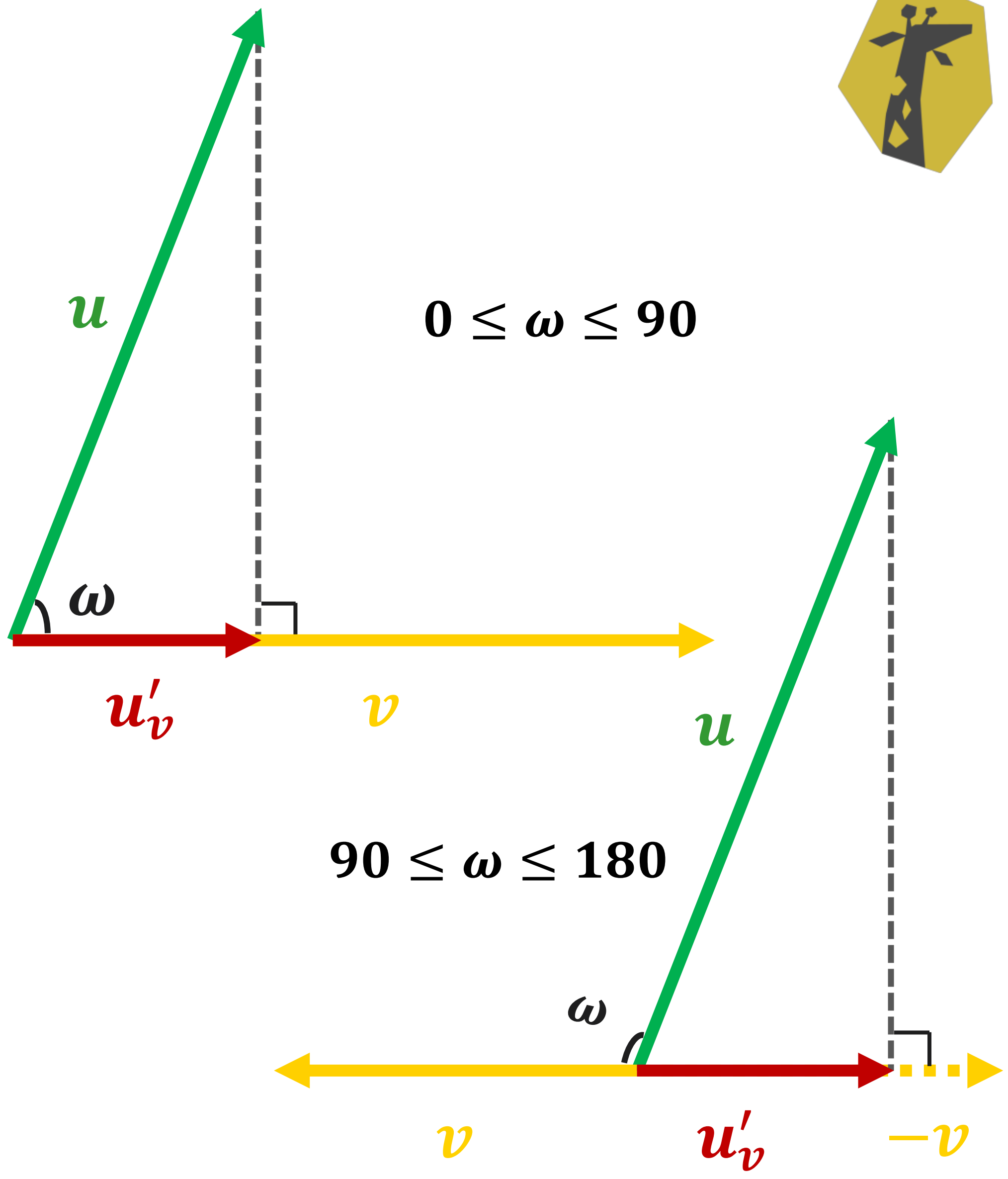
$$\boldsymbol{u}'_v = \frac{(\boldsymbol{u}, \boldsymbol{v})}{(\boldsymbol{v}, \boldsymbol{v})} \boldsymbol{v}.$$



Orthogonal Projection: Example

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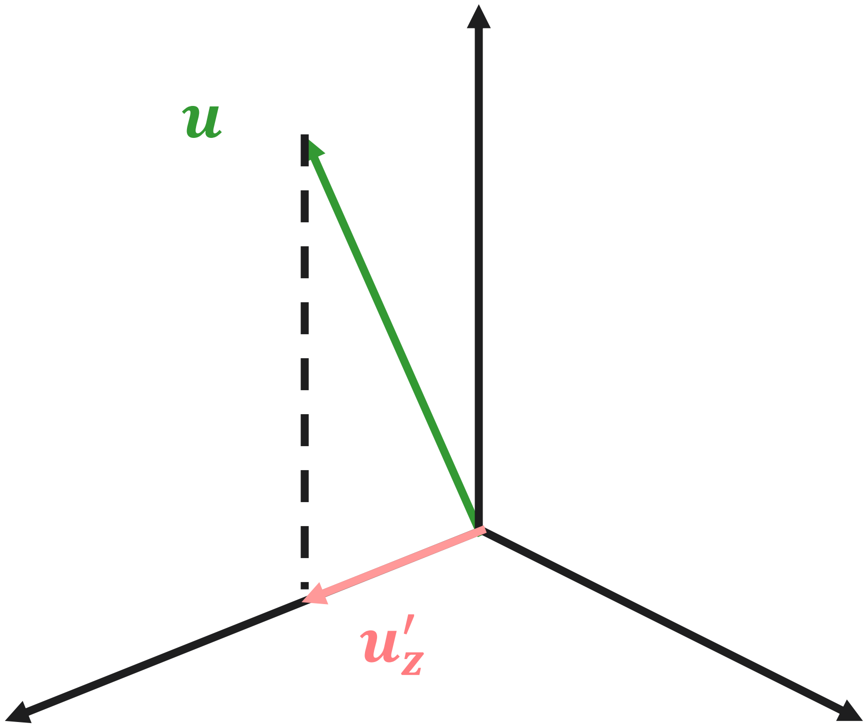

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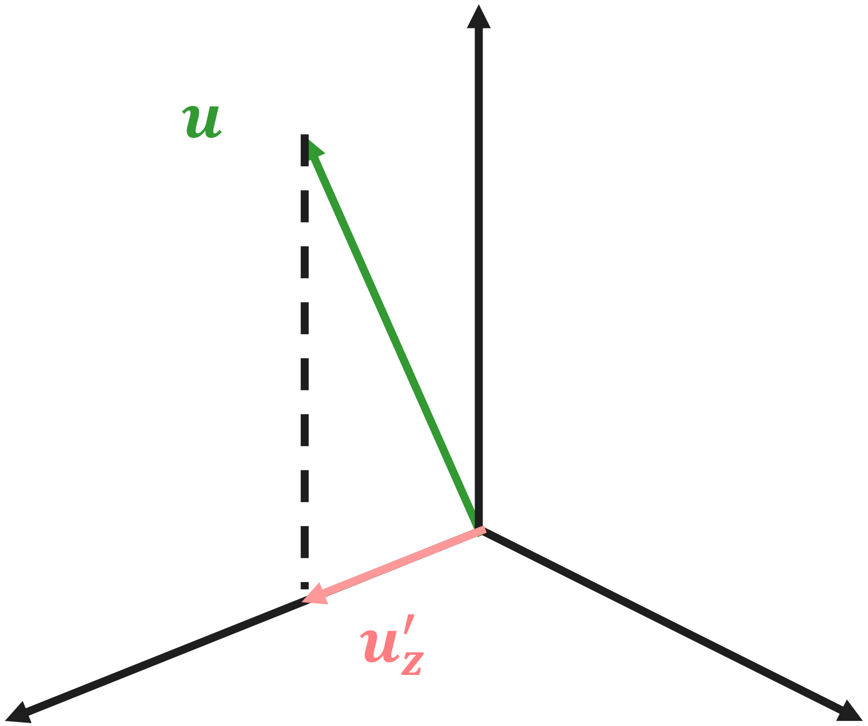

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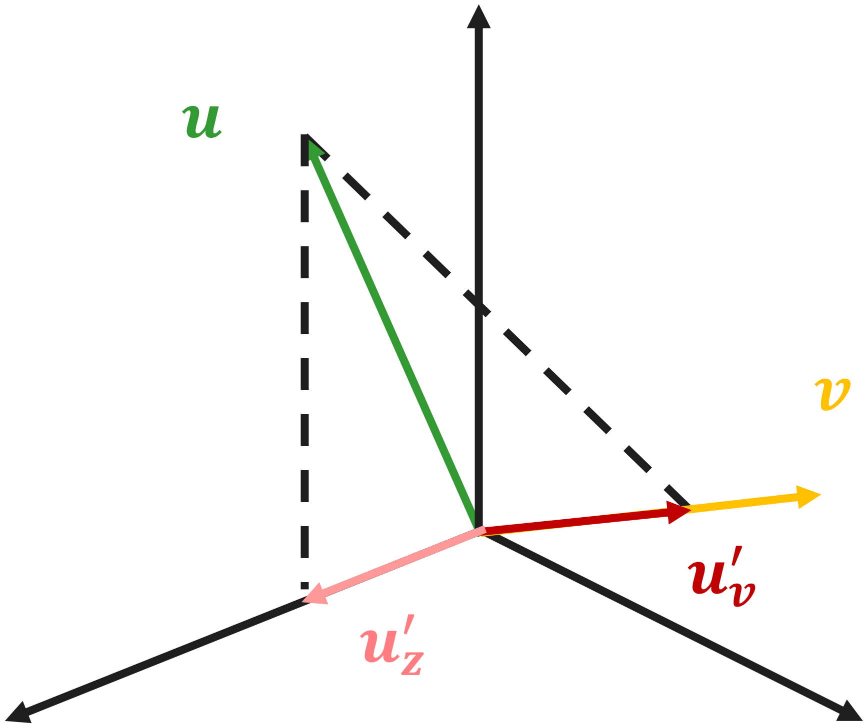

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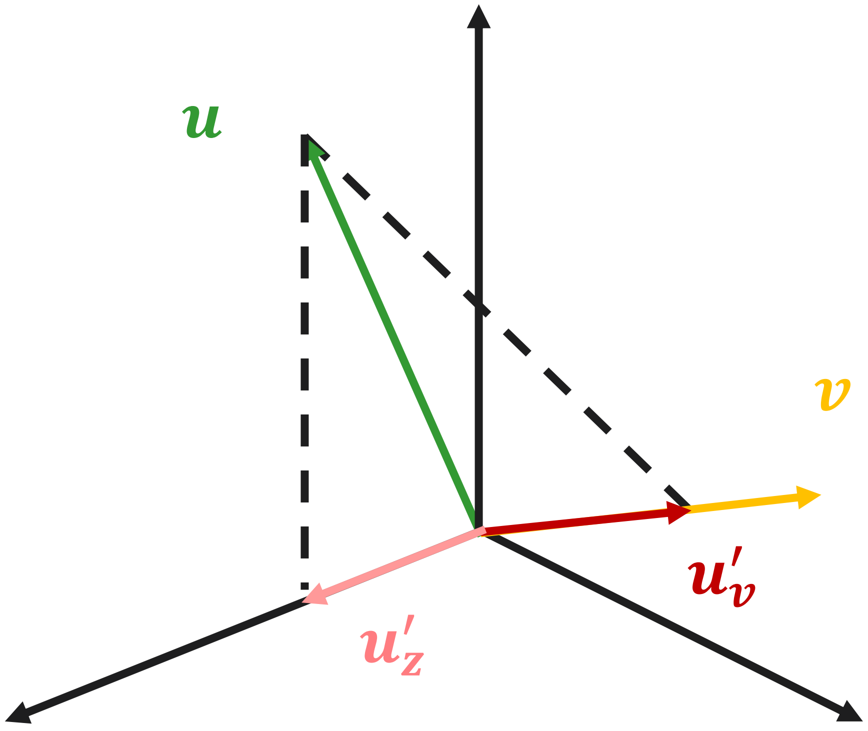
- What's projection of $u = [1, 3, 2]$ on $\mathrm{v} = [4, 1, 3]$?

$$u'_v = \frac{(u, v)}{(v, v)} v = \frac{4 + 3 + 6}{16 + 1 + 9} v = \frac{1}{2} v = [2, 0.5, 1.5].$$



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Hyperplanes

- A hyperplane is described by equation

$$w_1x_1 + w_2x_2 + \cdots + w_nx_n + b = 0$$

where at least one $w_i \neq 0$.

- A more compact notation:

$$(w, x) + b = 0, \qquad w = (w_1, \dots, w_n)$$

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Hyperplanes

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- A hyperplane in $\mathbb{R}^n$ is described by equation

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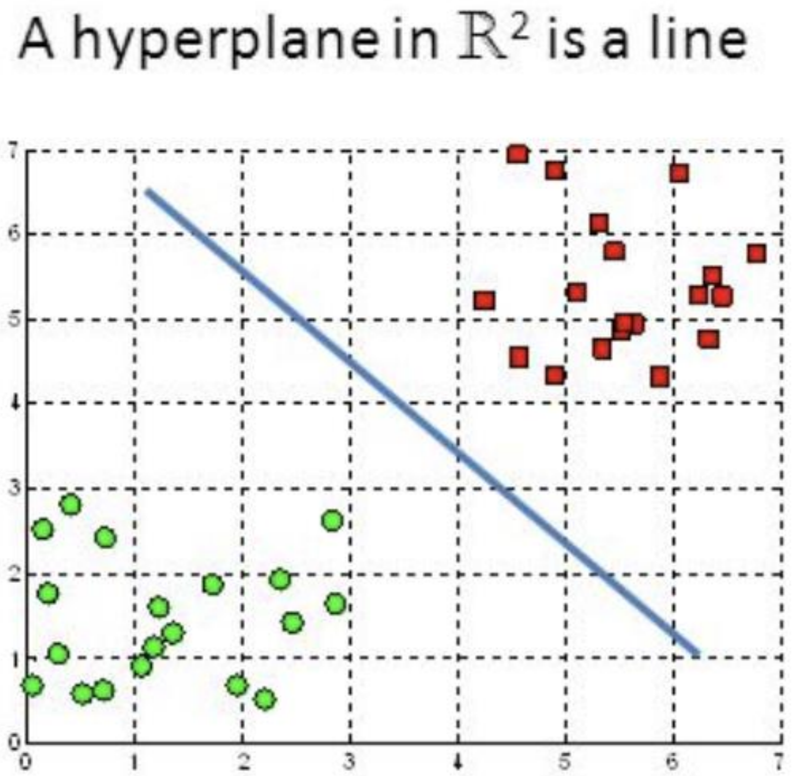
Hyperplanes



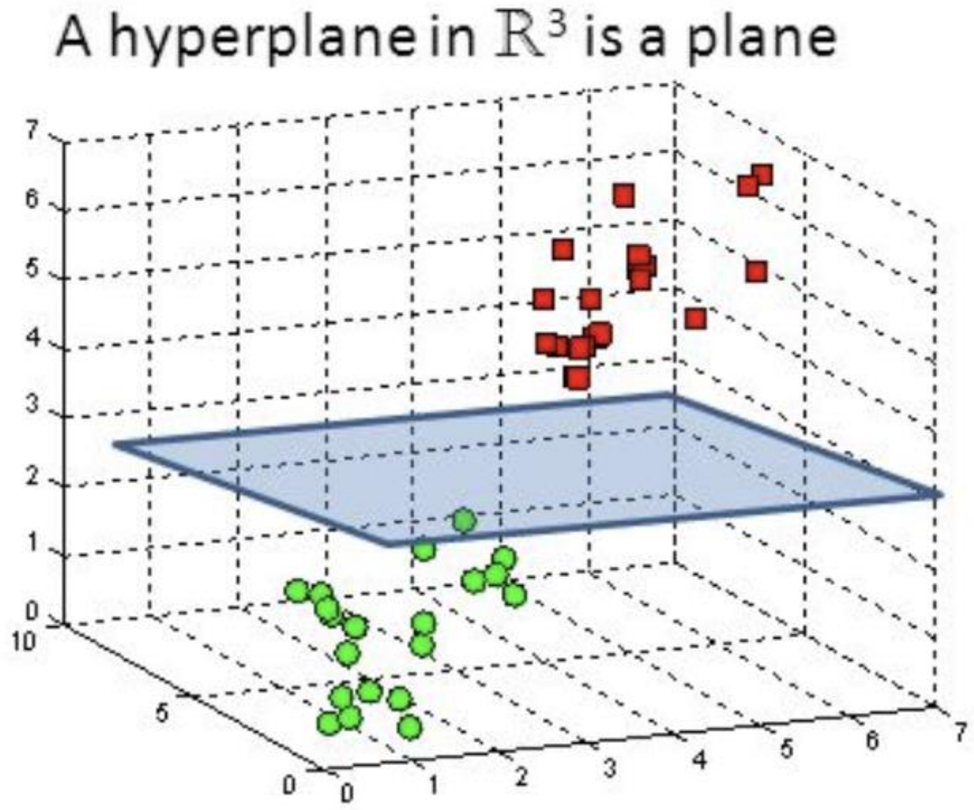


Image source: https://deepai.org/machine-learning-glossary-and-terms/hyperplane

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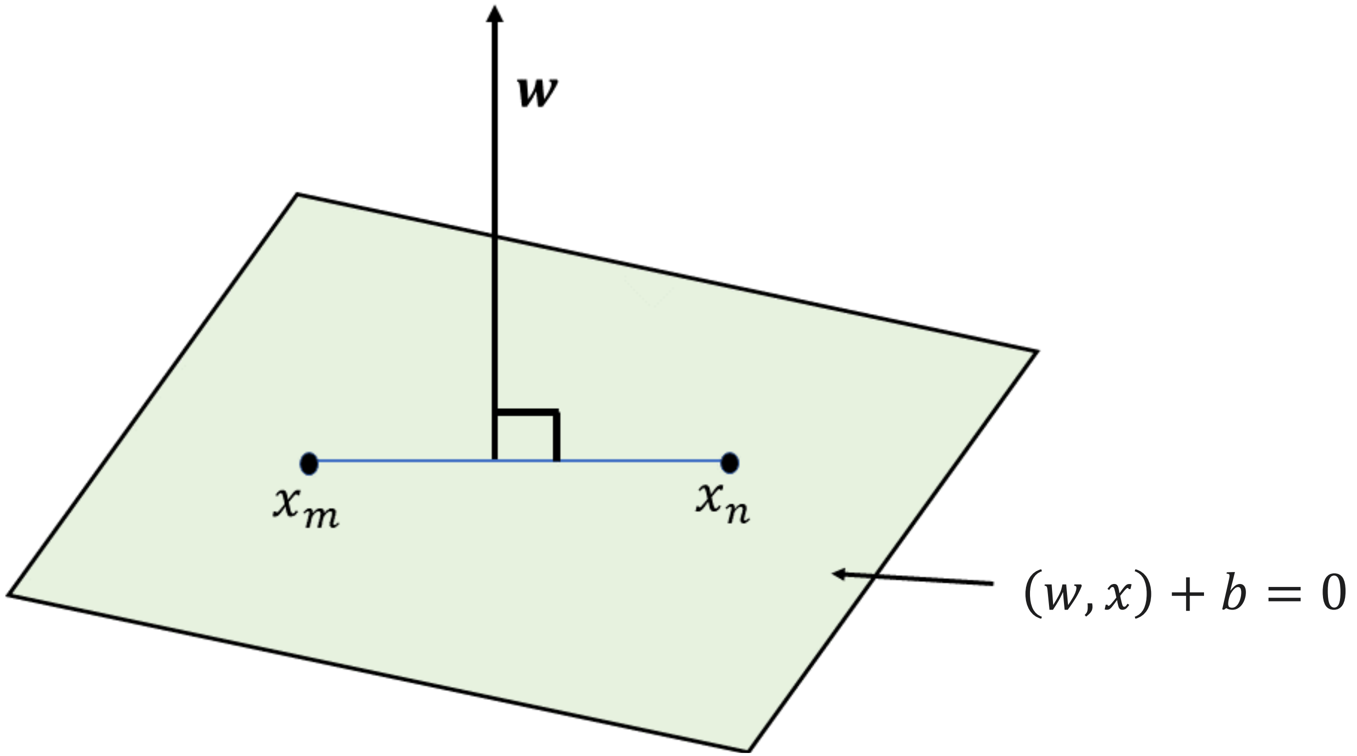
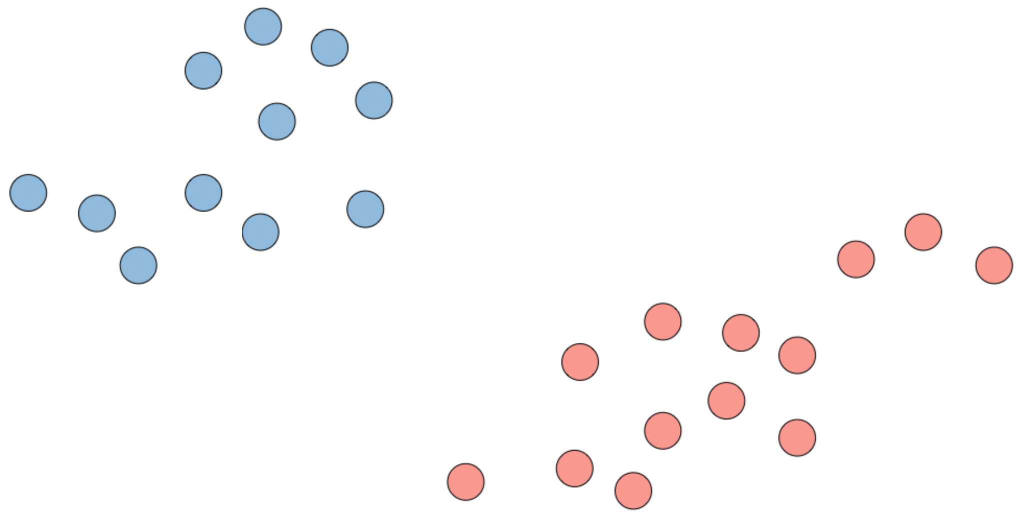
Normal to a Hyperplane

- Consider a hyperplane $(w, x) + b = 0$.
- Vector $w = (w_1, \dots, w_n)$ defines the hyperplane.
- w is a *normal vector* to this hyperplane: it's orthogonal to every vector on it.



ML Example: Linear Classifier

- Objects = 2D vectors
- Binary classification: classes +1 and -1.



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  - objects “above” are class +1
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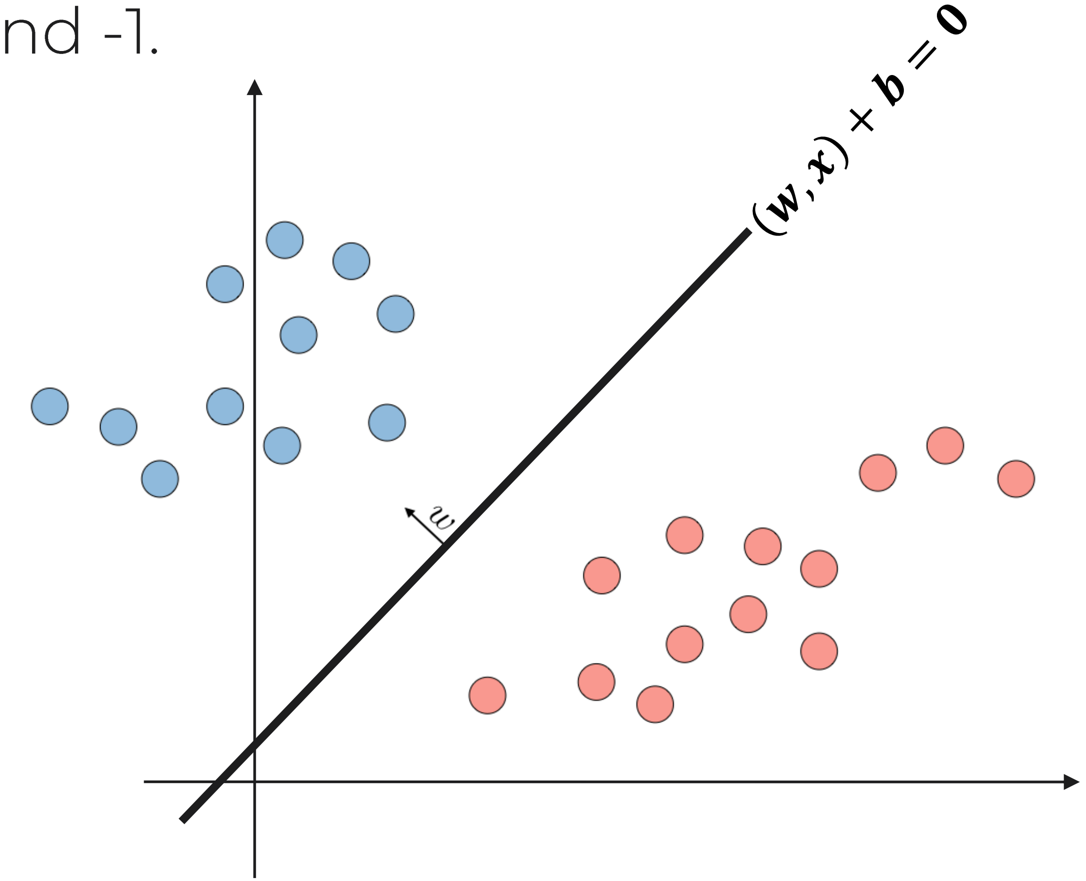

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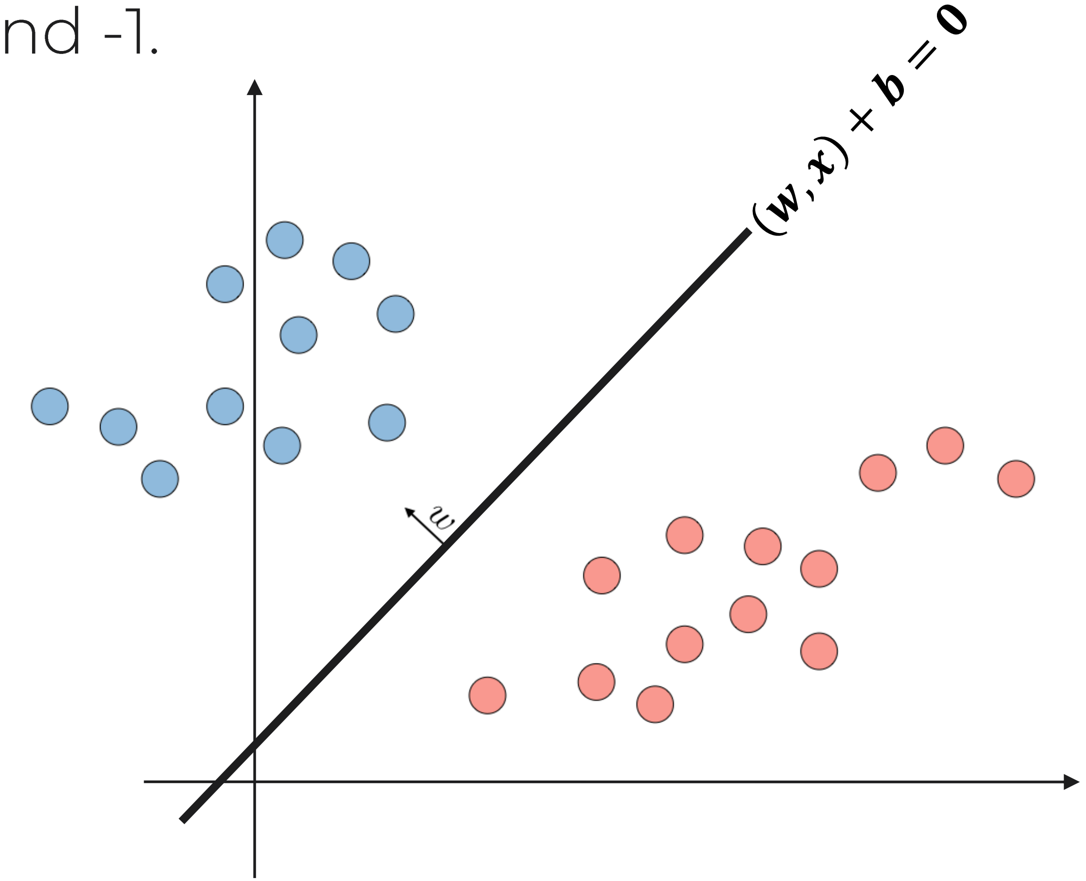

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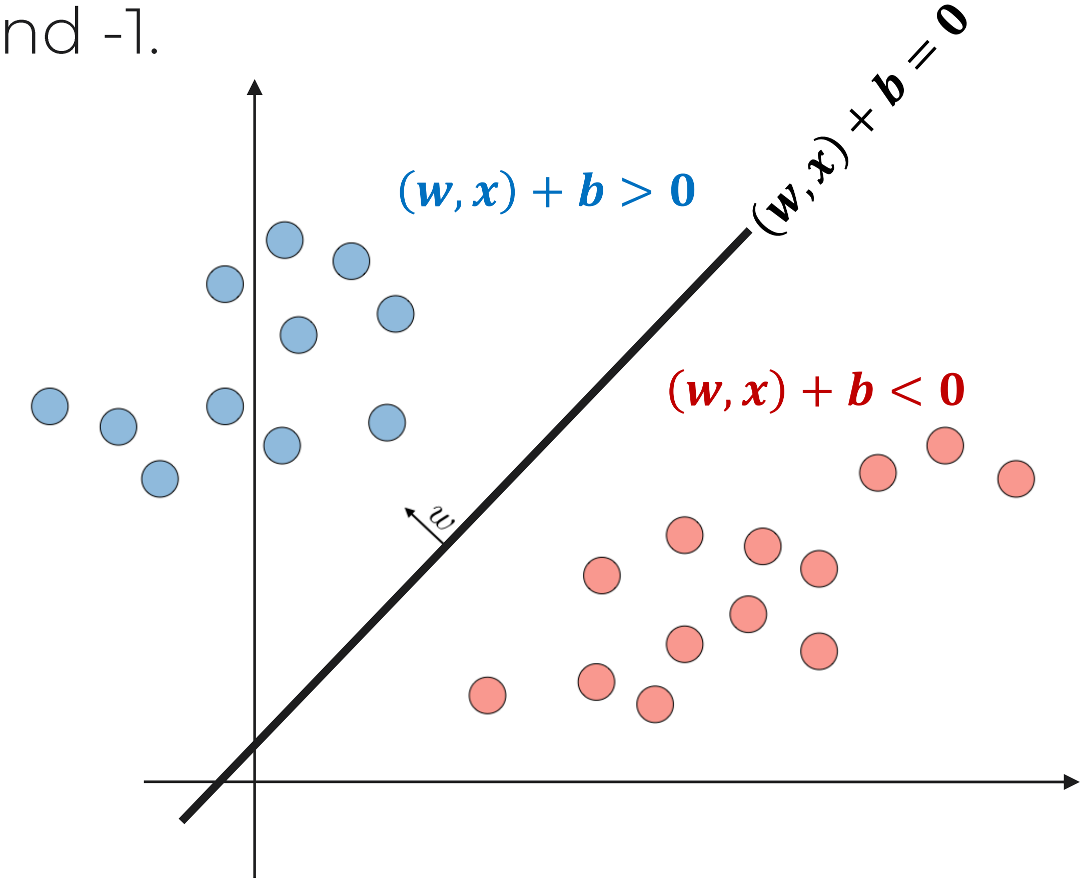
- How can we formalize this?
  - objects “above”: $(w, x) + b > 0$
  - objects “below”: $(w, x) + b < 0$



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To sum up

- Vectors
  - Vector spaces
  - Inner products
  - Lengths
  - Distances
  - Angles
- Analytic Geometry
  - Projections
  - Hyperplanes
  - Normal vector

Math Refresher for DS

Lecture 2

Last Time

- Vector spaces
- Euclidian spaces (= vector spaces + dot product)
- Length of a vector
- Distances and angles between the vectors
- Orthogonality
- Orthogonal projections

Today

- Back to vector spaces
  - Linear independence
  - Basis

- Basic operations with matrices.

Back to Vector Spaces

(Reminder) Vector Space: Definition

- A real-valued **vector space** $(V, +, \ \cdot)$ is a set of vectors V with two operations

$$(1)+: \ \ V \times V \to V, \qquad (2)\cdot: \ \ \mathbb{R} \times V \to V$$

that satisfy the following properties (axioms):

	Property	Meaning
1.	**Associativity** of addition	$x + (y + z) = (x + y) + z$
2.	**Commutativity** of addition	$x + y = y + x$
3.	**Identity element** of addition	$\exists 0 \in V: \ \forall x \in V \quad 0 + x = x$
4.	**Identity element** of scalar multiplication	$\forall x \in V \quad 1 \cdot x = x$
5.	**Inverse element** of addition	$\forall x \in V \ \exists -x \in V: \ \ x + (-x) = 0$
6.	**Compatibility** of scalar multiplication	$\alpha(\beta x) = (\alpha\beta)x$
7.	**Distributivity**	$(\alpha + \beta)x = \alpha x + \beta x$
8.		$\alpha(x + y) = \alpha x + \alpha y$

(Reminder) Examples of Vector Spaces

- $\mathbb{R}^n$ - a set of vectors with n real entries.
  $(\mathbb{R}^n, +, \cdot)$ is a vector space.

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- $\mathbb{R}^n$ - a set of vectors with n real entries.
  $(\mathbb{R}^n, +, \cdot)$ is a vector space.

- $\mathbb{P}^n$ - a set of polynomials of degree $\leq n$ with real coefficients
  $(\mathbb{P}^n,\ +,\ \cdot)$ is also a vector space!
  "Vectors" here are polynomials.

Vector Subspaces

Vector Subspace

- $V = (\mathbb{V}, +, \cdot)$ - a vector space.

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Vector Subspace

- $V = (\mathbb{V},\ +,\ \cdot)$ - a vector space.
- Consider $\mathbb{U} \neq \emptyset$ – a subset of $\mathbb{V}$ $(\mathbb{U} \subseteq \mathbb{V})$.
- $U = (\mathbb{U},\ +,\ \cdot)$ - *a vector subspace* $(U \subseteq V)$ if U is a vector space with operations
    - $+ : \mathbb{U} \times \mathbb{U} \to \mathbb{U}$
    - $\cdot\ : \mathbb{R} \times \mathbb{U} \to \mathbb{U}$

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Vector Subspace

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- In fact, we only need to check:
  1. that $\mathbf{0} \in \mathbb{U}$
  2. closure of $+$ and $\cdot$:
     - $\forall\, x, y \in \mathbb{U}\ \ x + y \in \mathbb{U}$
     - $\forall\, x \in \mathbb{U},\ \lambda \in \mathbb{R}\ \ \lambda x \in \mathbb{U}$

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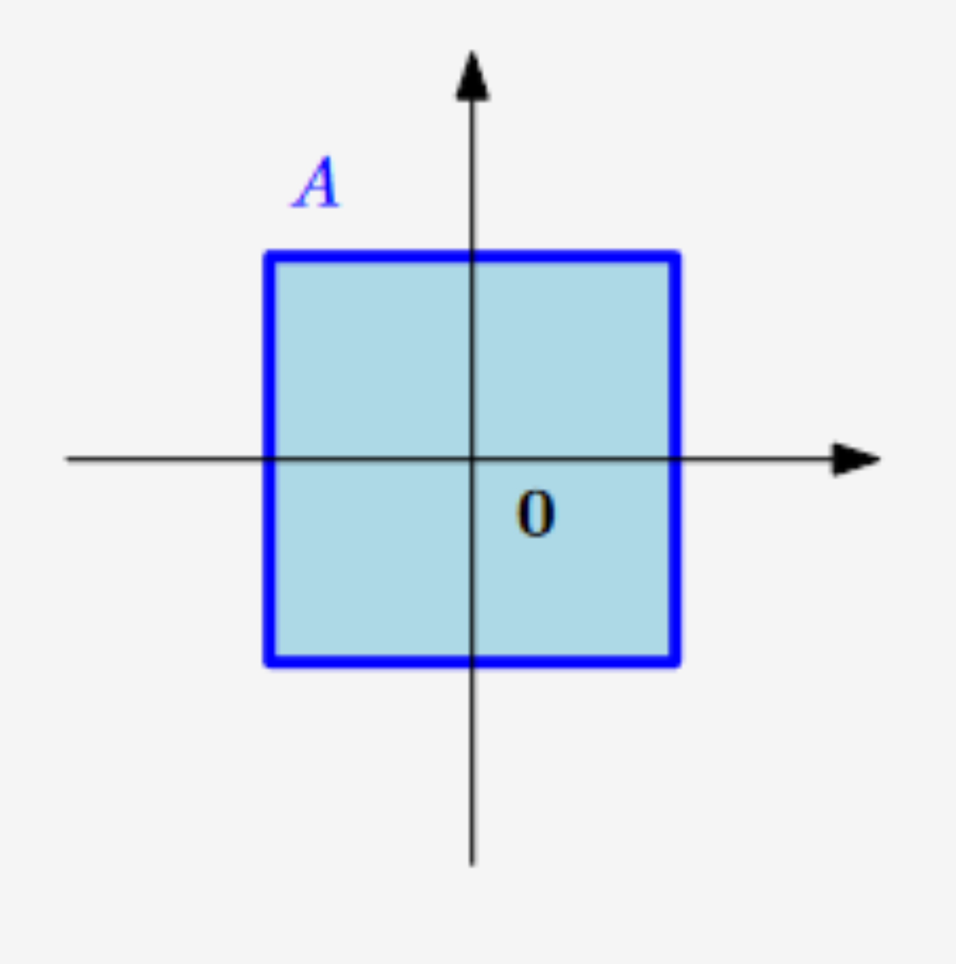
Vector Subspace: Examples

- Consider $\mathbb{R}^2$.
- Is $\boldsymbol{A}$ a vector subspace?



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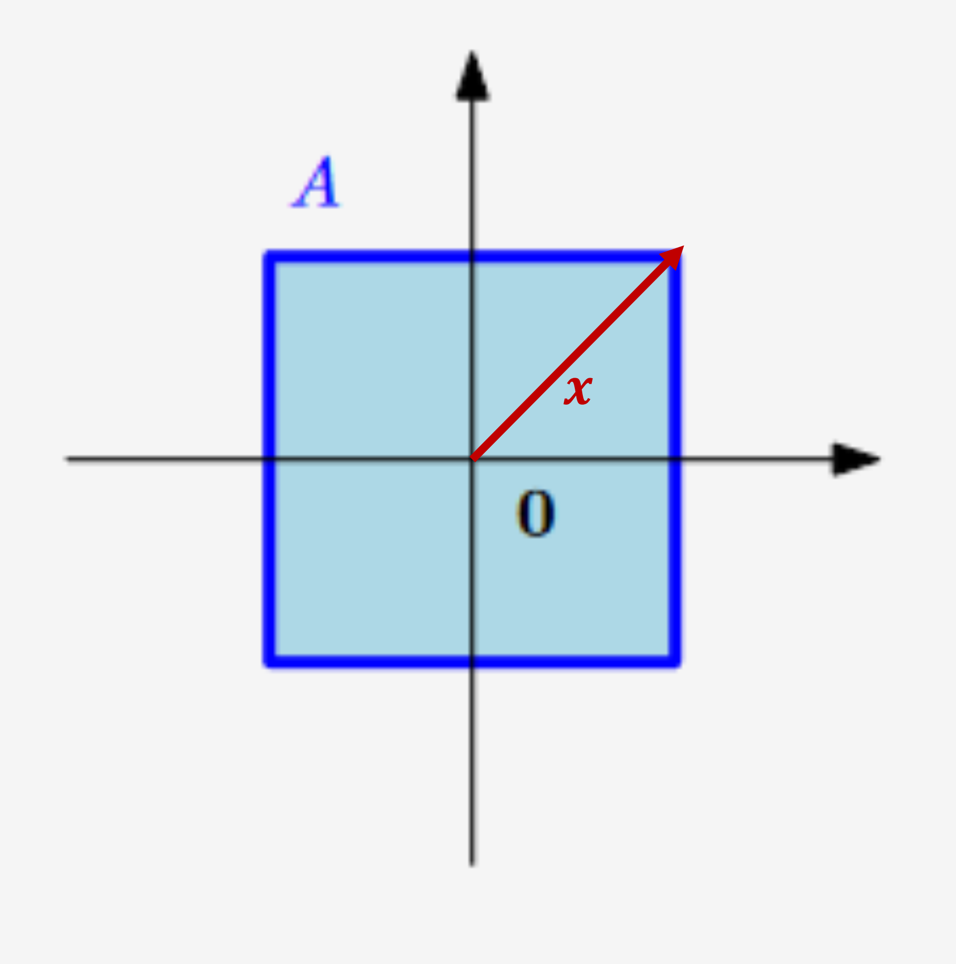

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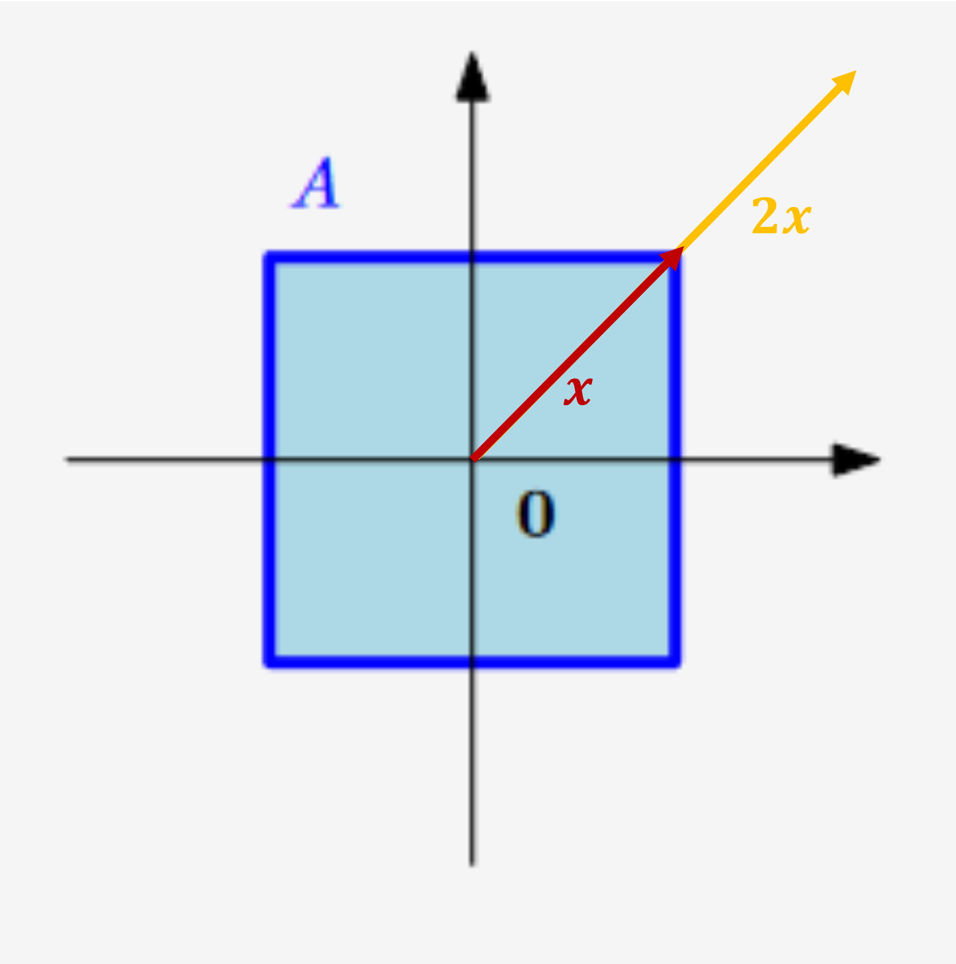
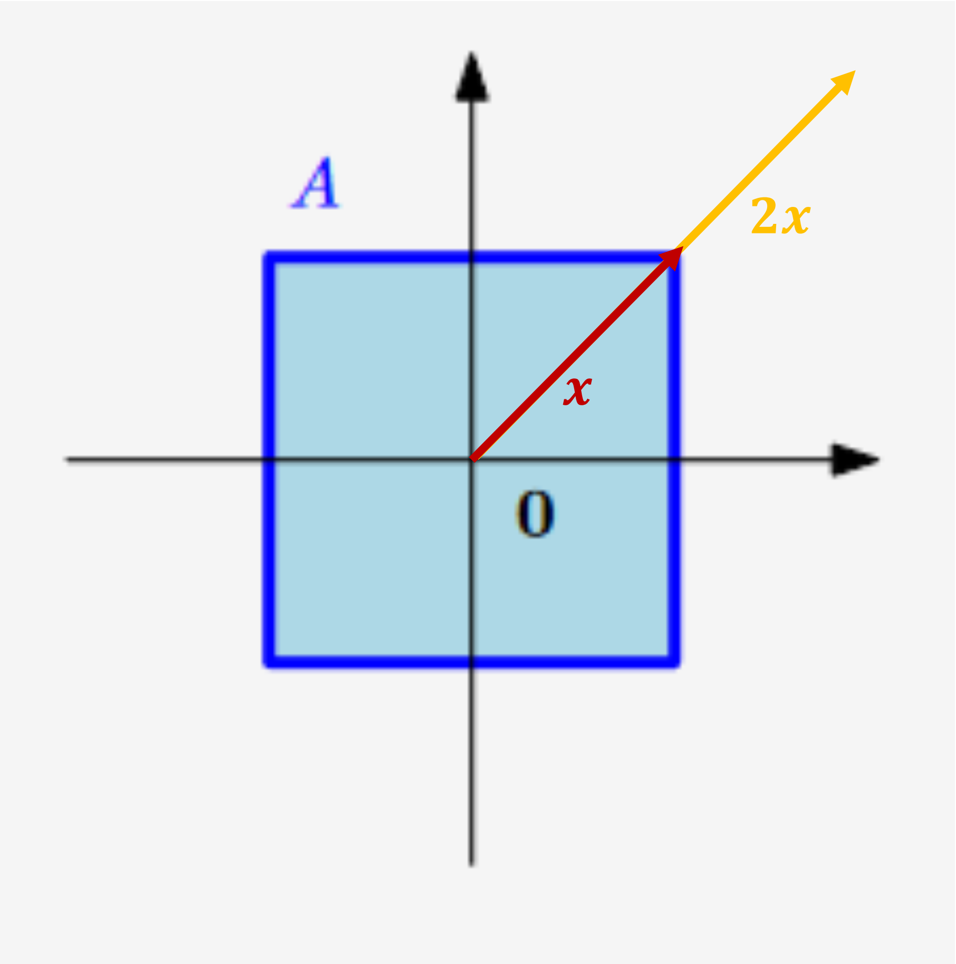


Vector Subspace: Examples

- Consider $\mathbb{R}^2$.
- Is $\boldsymbol{A}$ a vector subspace?
- No!

$\boldsymbol{x} \in \boldsymbol{A}$ but $\mathbf{2}\boldsymbol{x} \notin \boldsymbol{A} \rightarrow$
· operation isn't closed.



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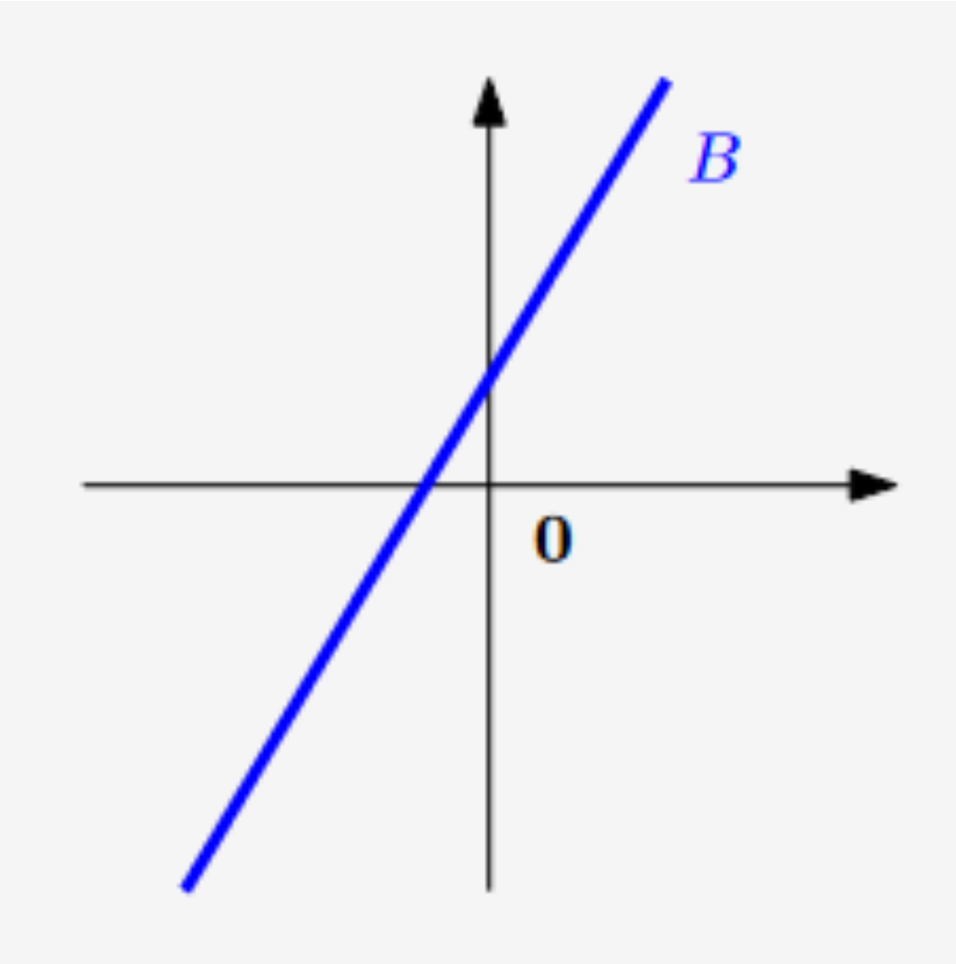

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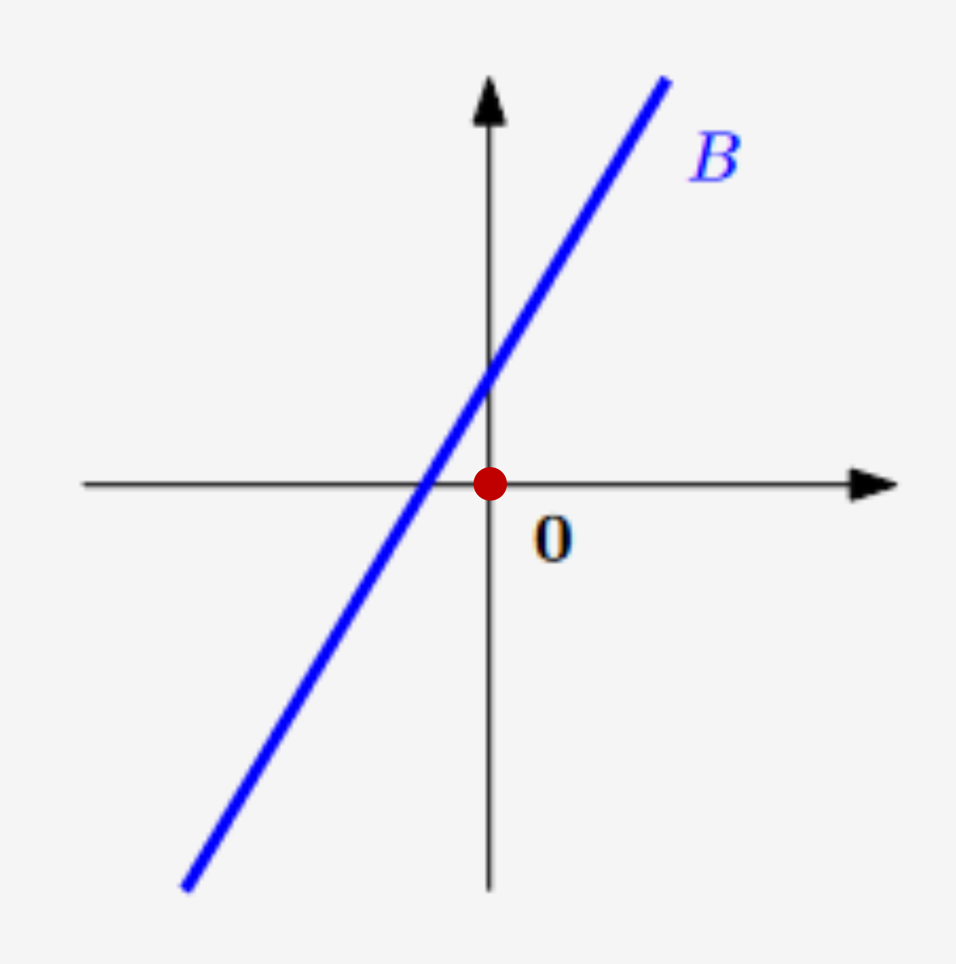
$$0 \notin B$$



Vector Subspace: Examples

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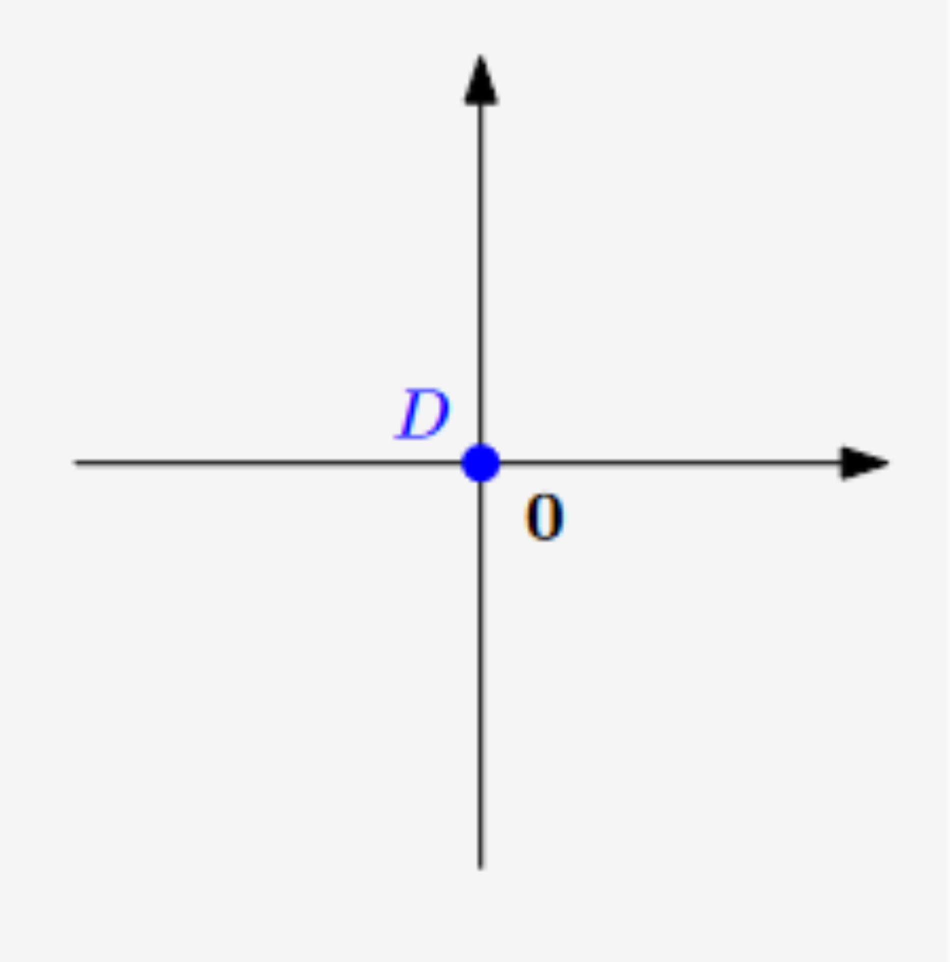

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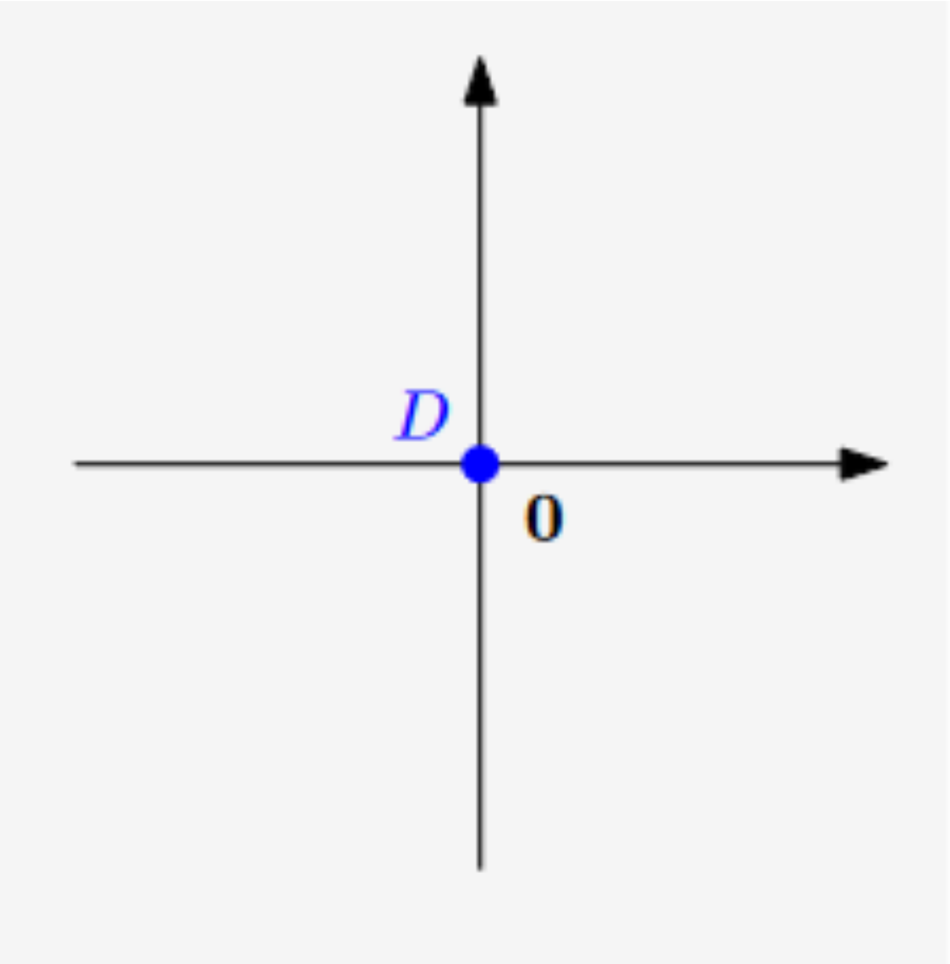

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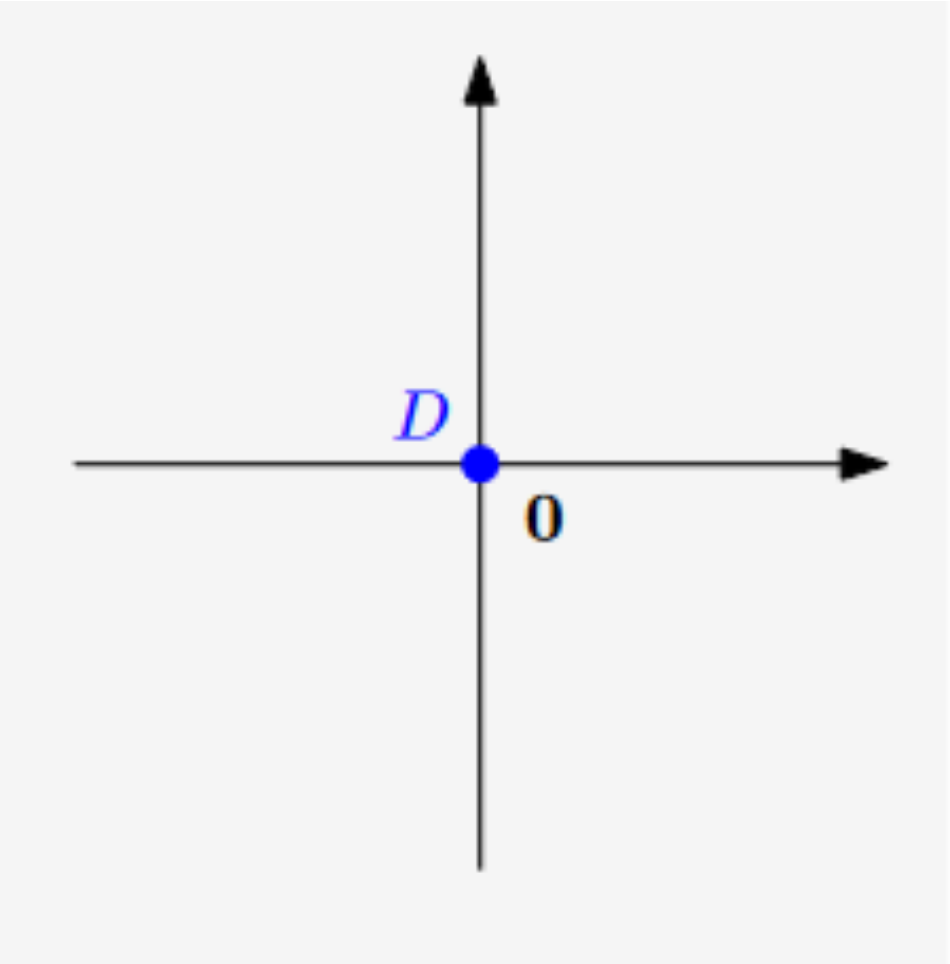
$\{0, +, \cdot\}$ *is a trivial vector subspace of any vector space*



Vector Subspace: Examples

- Consider $P^n = (\mathbb{P}^n, +, \ \cdot)$ - a vector space.

Vector Subspace: Examples

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  - Closure: when we add up polynomials of degree $m \leq n$ or multiply them by a scalar, we always get a polynomial of of degree $m \leq n$.

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$v = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k = \sum_{i=1}^{k} \lambda_i x_i \in V$ –
a *linear combination* of $x_1, x_2, \dots, x_k$.

Linear Combinations: Examples

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  $v = 3t + 3 = 3e_1 + 3e_0$ is a linear combination of e_0 and e_1.

Span

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- If $\mathbb{A}$ spans vector space V, we write

$$V = span[\mathbb{A}] \text{ or } V = span[x_1, \dots, x_n].$$

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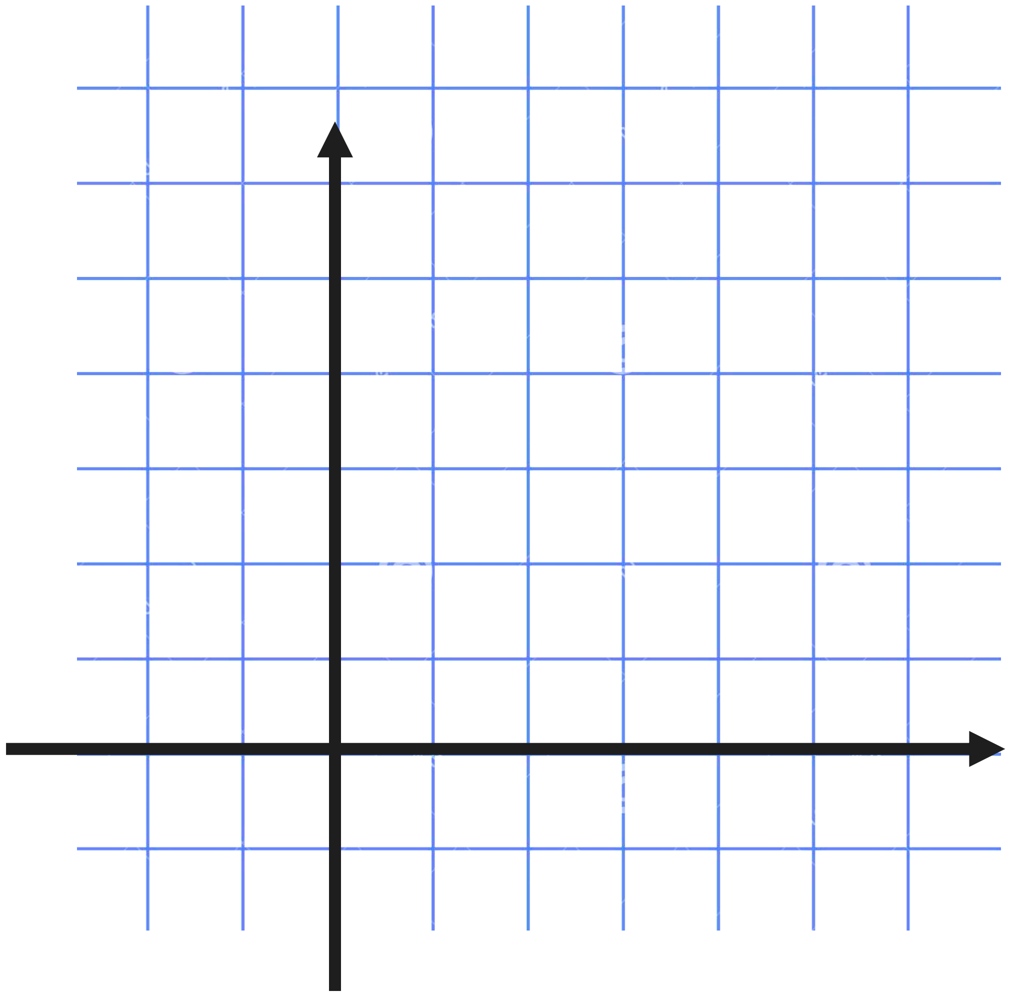
Span: Example 1

- Consider $\mathbb{R}^2$.



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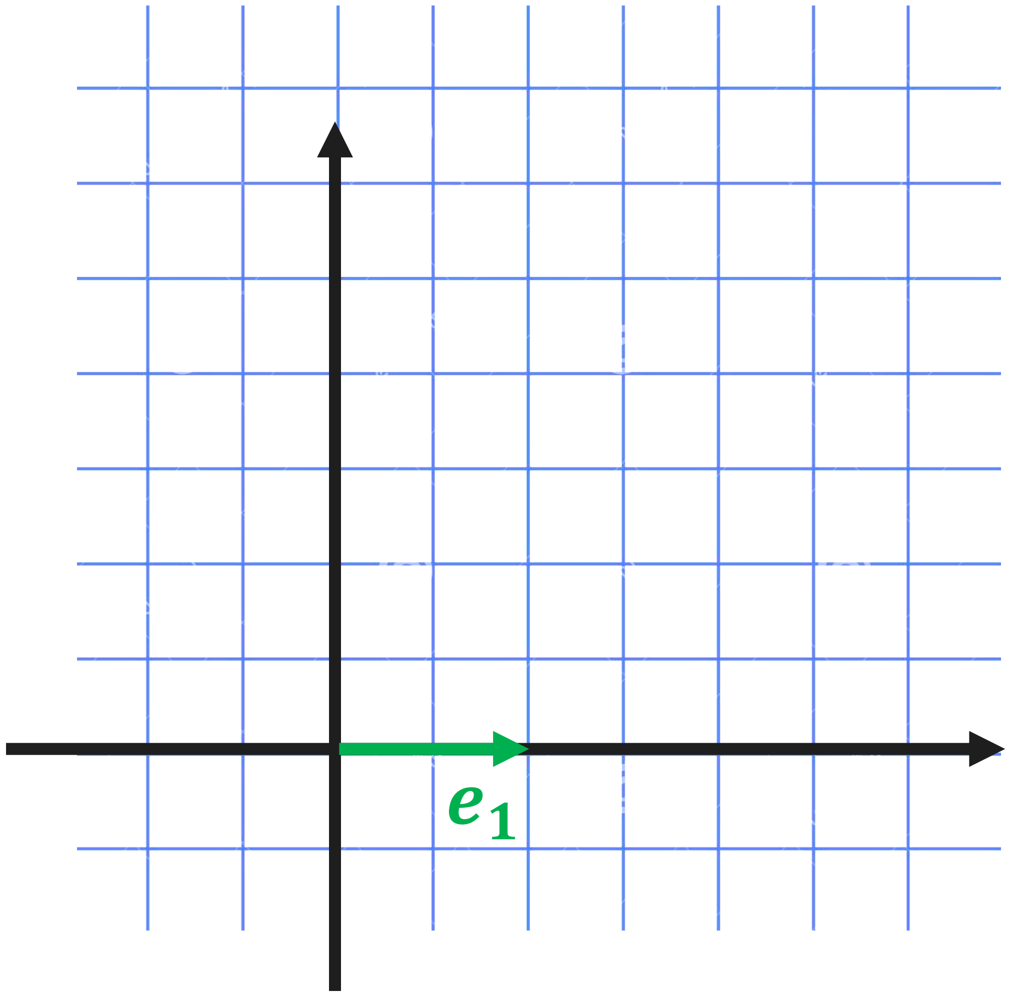

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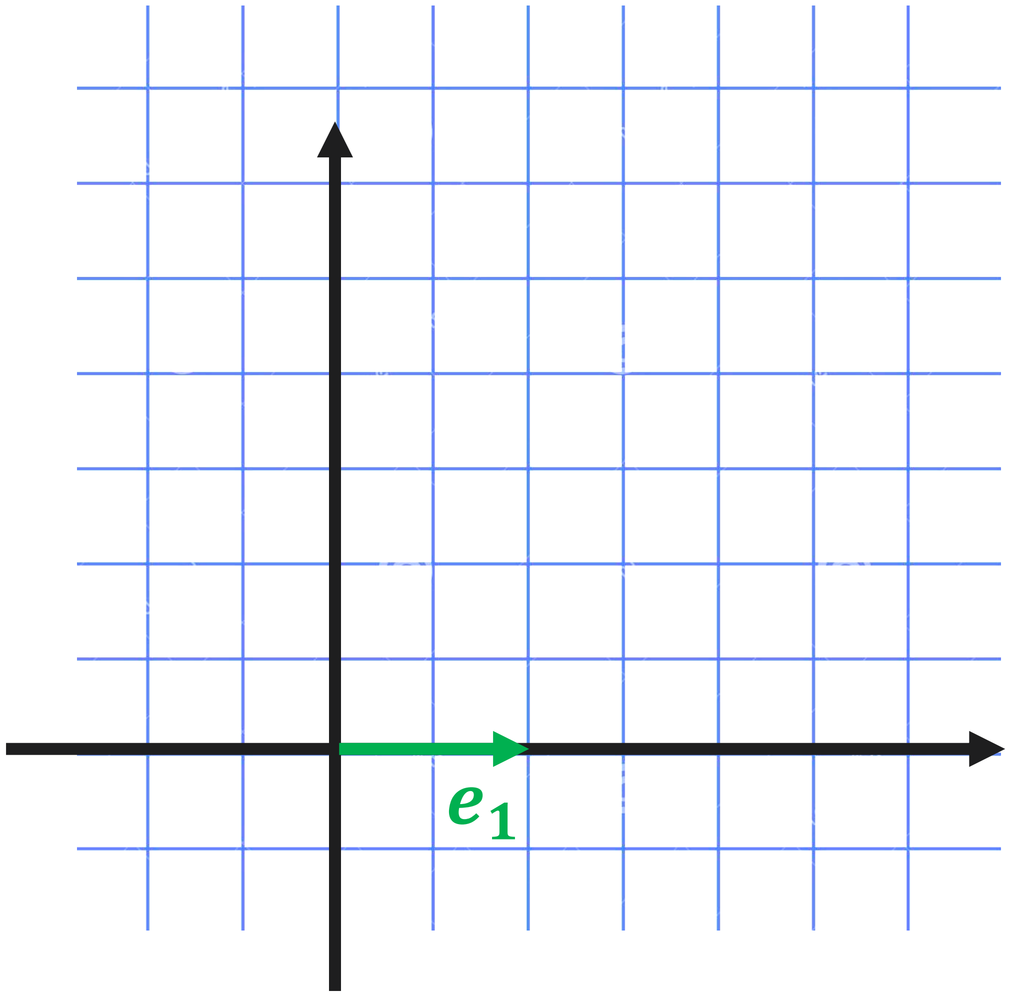

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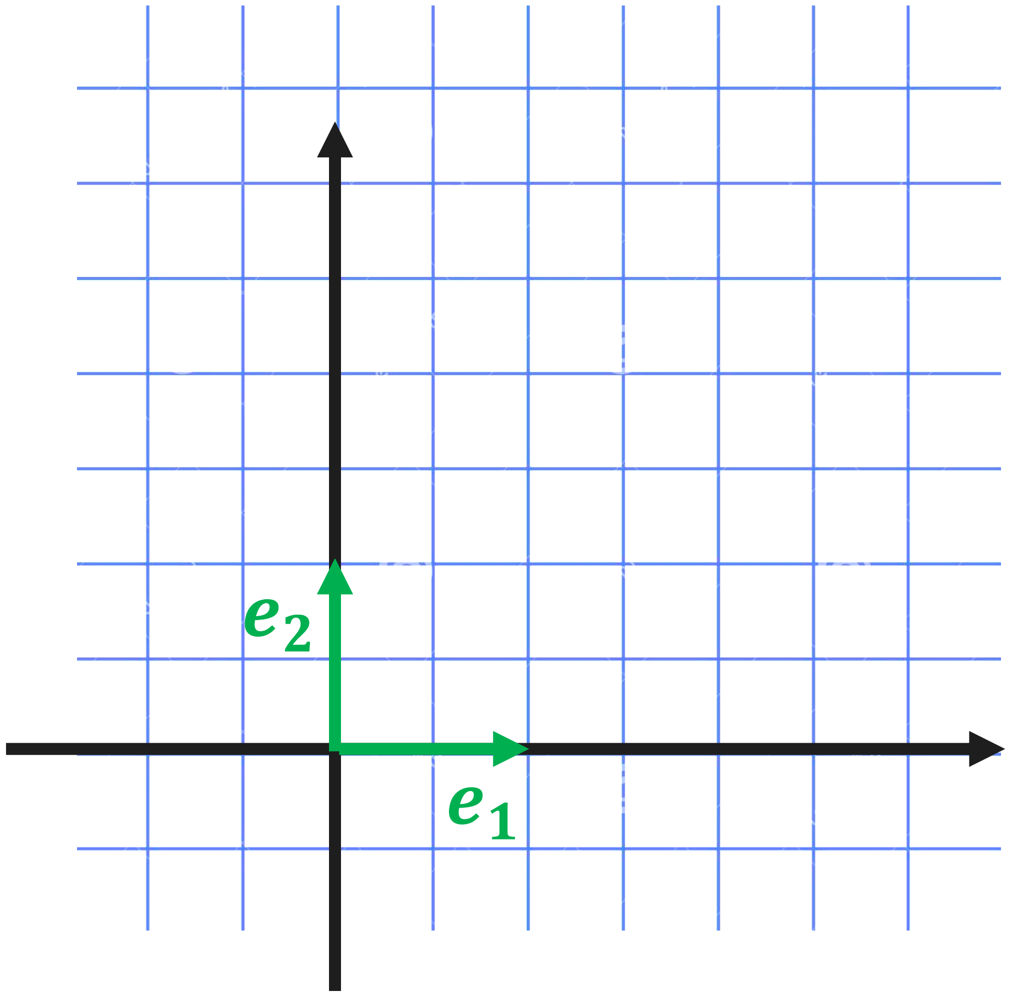

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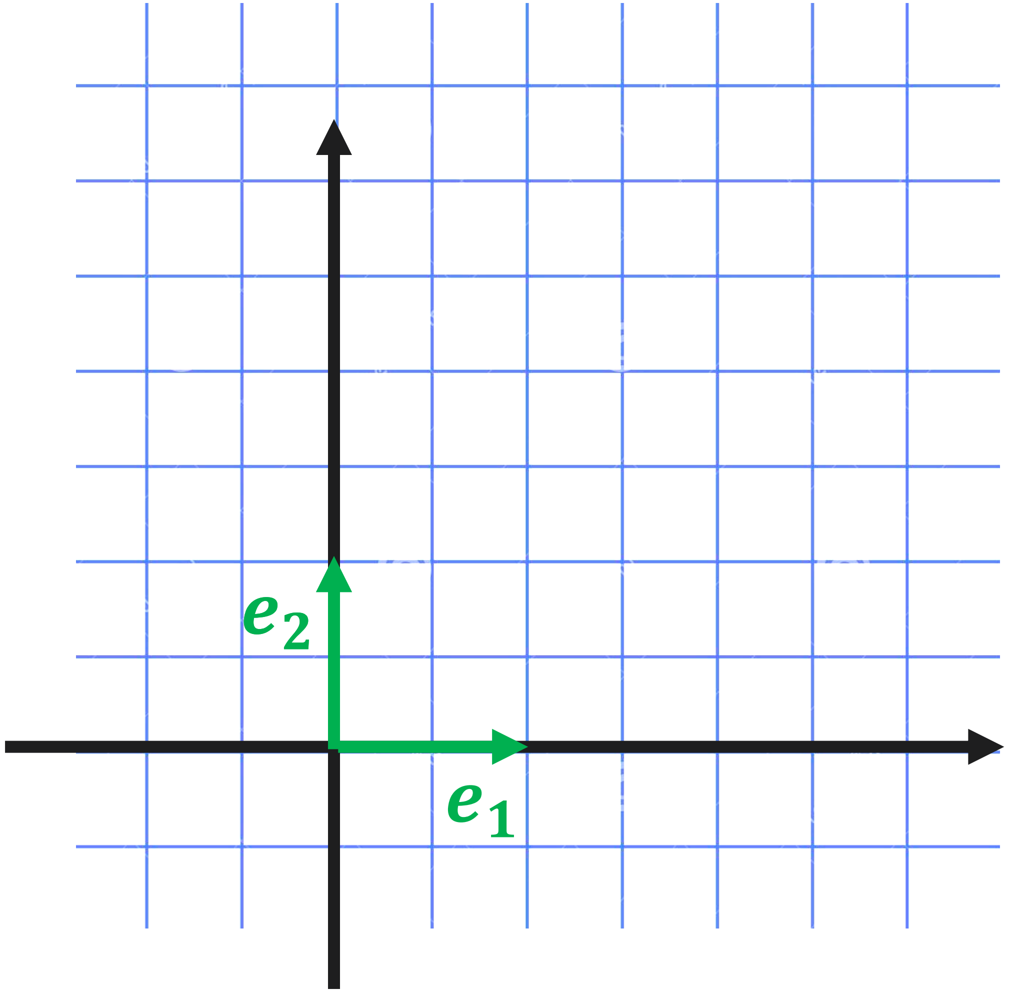

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Generating Set

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- If every vector $\mathbf{v} \in V$ can be expressed as a linear combination of $x_1, x_2, \dots, x_k$, $\mathbb{A}$ is called a *generating set* for V.

Linear independence

Linear Combinations

- A zero vector can always be represented as a trivial linear combination of $x_1, x_2, \dots, x_k$:

$$0 = \sum_{i=1}^{k} 0 \cdot x_i$$

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- We are mostly interested in *non-trivial linear combinations* of $x_1, x_2, \dots, x_k$ where not all λ_i are $\mathbf{0}$.

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- If only trivial solution exists, vectors $x_1, x_2, \dots, x_k$ are *linearly independent.*

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- A set of vectors $x_1, x_2, \dots, x_k$ is linearly dependent if and only if (at least) one of the vectors is a linear combination of the others

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Why is this so? Try to prove this yourself.

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  there are no $\lambda_1, \lambda_2 \in \mathbb{R}$ with at least one $\lambda_i \neq 0$ such that

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  (Or: you cannot represent $\boldsymbol{e}_1$ as $\lambda \boldsymbol{e}_2$ or vice versa).

Linear (In)dependence: Example 2

- Consider $P = (\mathbb{P}^3, +, \ \cdot)$.
- $1, \ t, \ t^2 \in P$ – vectors. Are they linearly independent?

Linear (In)dependence: Example 2

- Consider $P = (\mathbb{P}^3, +, \cdot)$.
- $1,\ t,\ t^2 \in P$ – vectors. Are they linearly independent?
- Yes!

  There is no way we can represent one of those vectors as a linear combination of the others.

Linear (In)dependence: Example 3

- Are the following vectors in $\mathbb{R}^4$ linearly independent?

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

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$$\lambda_1 \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1 + \lambda_2 - \lambda_3 \\ 2\lambda_1 + \lambda_2 - 2\lambda_3 \\ -3\lambda_1 + \lambda_3 \\ 4\lambda_1 + 2\lambda_2 + \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Next lecture: a better way to solve such systems of equations

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Dimension of a Linear Space

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  We denote this as $\mathbf{dim}(V) = n$.

Dimension: Example

- $\mathbb{R}^n$ is a n-dimensional vector space. Why?
- Consider n vectors $e_1, \dots, e_n$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

- $e_1, \dots, e_n$ are linearly independent → $\mathbf{dim}(\mathbb{R}^n) \geq n$.
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  No!
  Explanation: next lecture.

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$\rightarrow \dim(P^3) = 4.$

Basis

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- Basis is A set of vectors with which we can represent every vector in the vector space by adding them together and scaling them.

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- $e_0 = 0, \ e_1 = t, \ \dots, e_n = t^n$ is a basis for P^n.

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Basis: Example

- Find the basis of a vector space spanned by vectors

$$x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad y = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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- Therefore, $V = span[x, y, z] = span[x, y]$.
  $B = \{x, y\}$ - basis of V.

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$$v = a_1 e_1 + a_2 e_2 + \dots + a_n e_n$$

- $a_1, a_2, \dots, a_n$ - *coordinates* of the vector v in the basis $e_1, e_2, \dots, e_n$.

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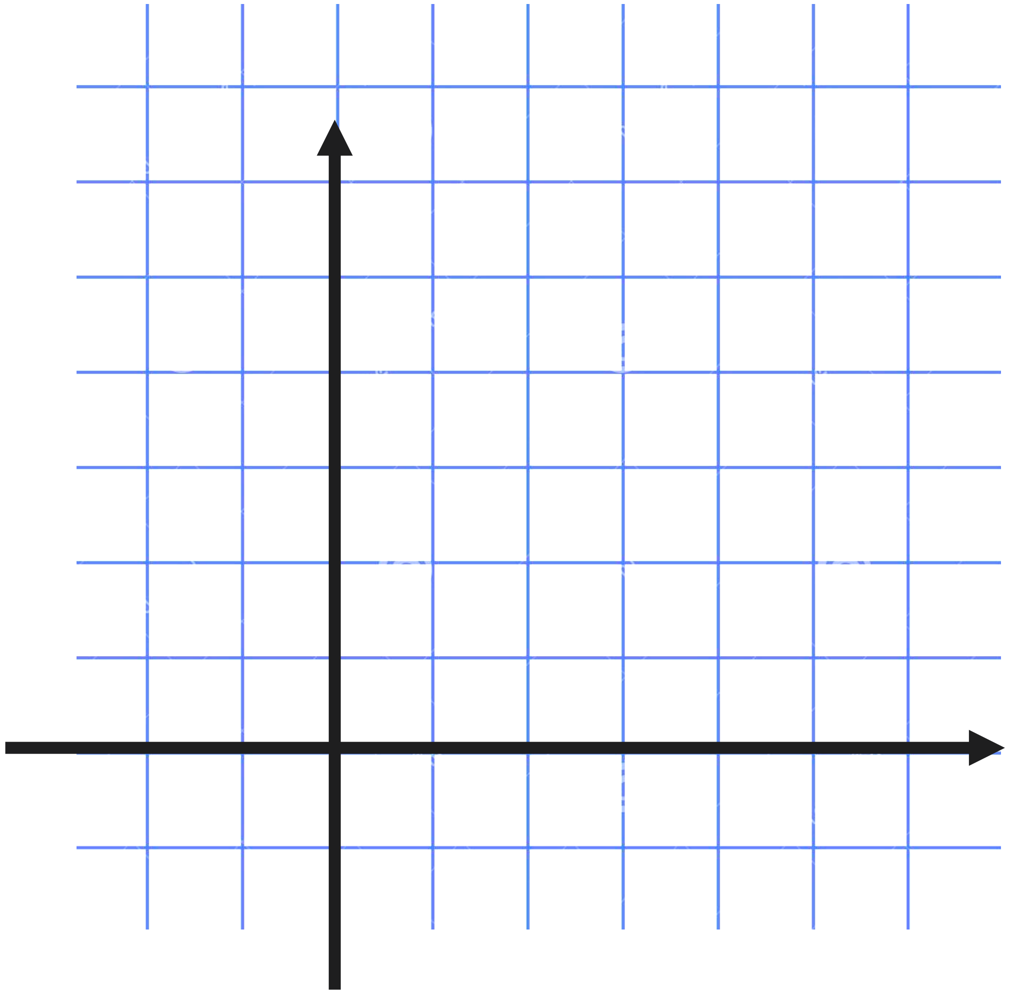
Coordinates: Example

- Consider $\mathbb{R}^2$.



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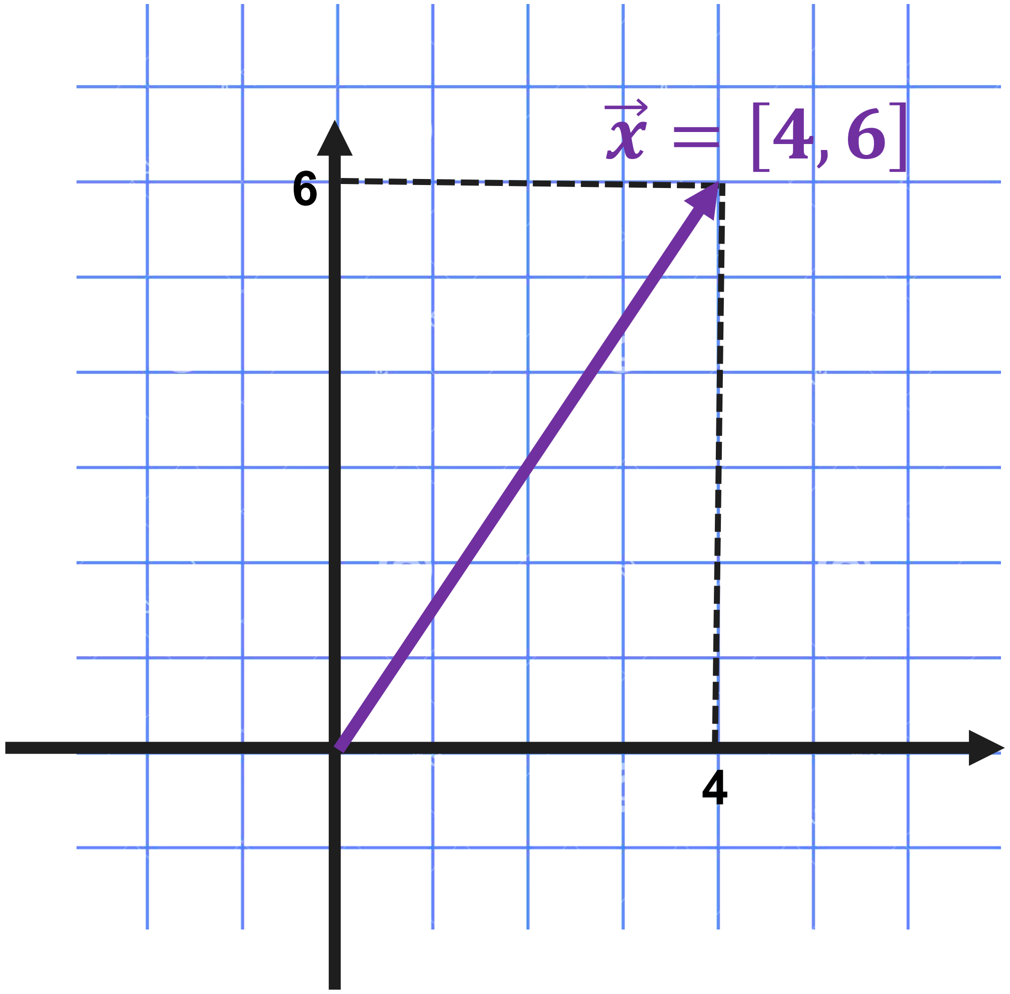

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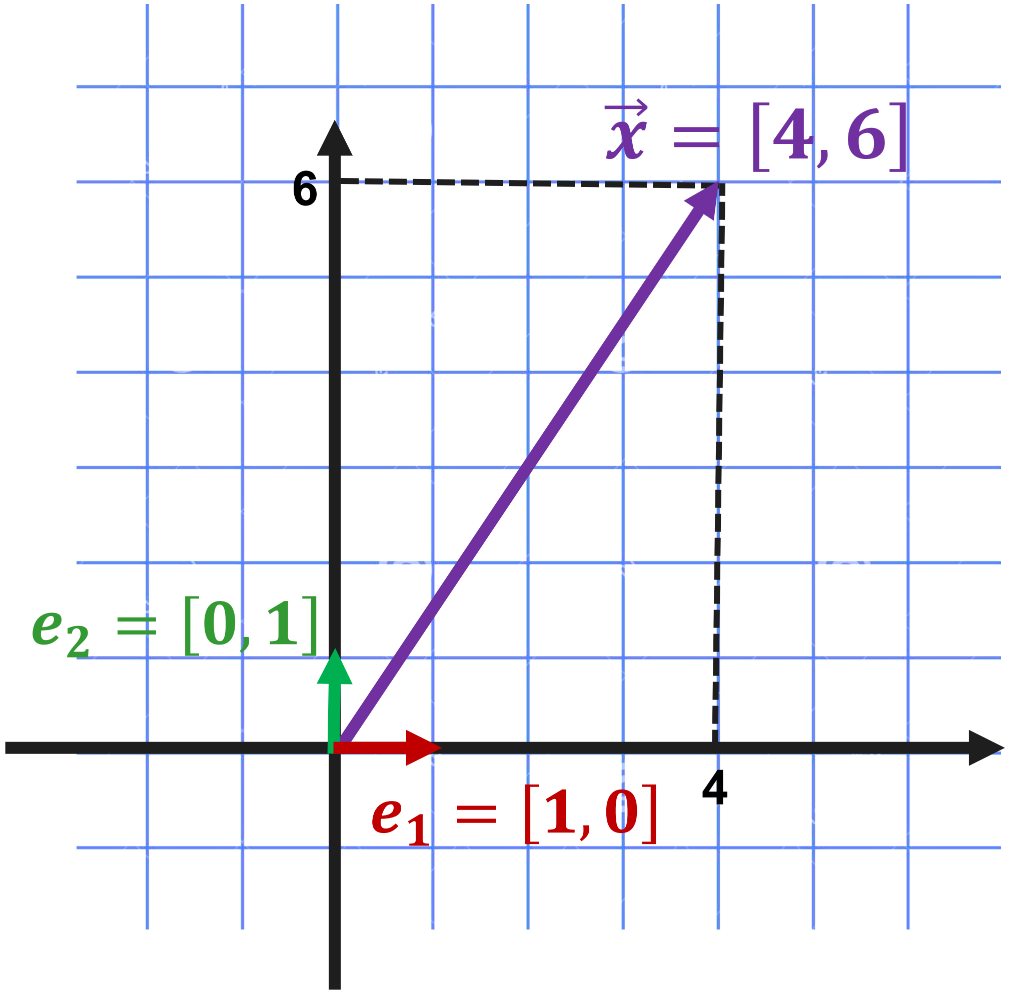

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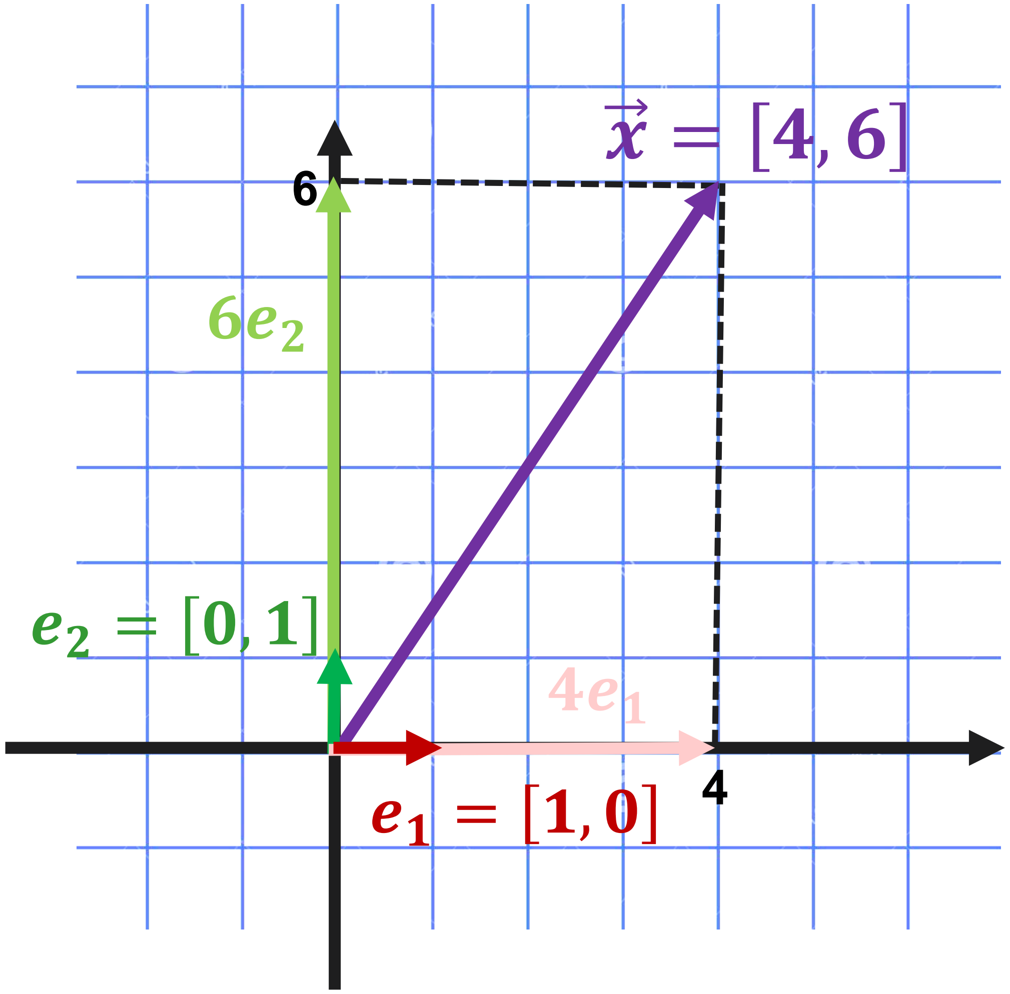

Orthogonal Basis

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- *Gram-Schmidt process*: a way to convert any basis to an orthogonal one. More details: practical session.

Change of Basis

Basis

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    - $b = \left\{b_1 = \begin{bmatrix}2\\1\end{bmatrix}, b_2 = \begin{bmatrix}-1\\-1\end{bmatrix}\right\}$ - yet another one.
- Different basis = different coordinates.
  How exactly do they change?

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with canonical basis

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- What are the coordinates in the new basis?

$$x_{new} = ?$$

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Coordinate Change: General Case

- Consider a vector space V with basis $e_1, e_2, \ldots e_n$.

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- New basis: $e'_1, e'_2, \dots e'_n$.

- What are the coordinates of x in this new basis?

$$x'_1, x'_2, \dots, x'_n = ?$$

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Coordinate Change

- Old basis: $e_1, e_2, \dots e_n$
  New basis: $e'_1, e'_2, \dots e'_n$

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- Coordinates of the new basis in the old one:

Coordinate Change

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$$
\begin{gathered}
e'_1 = \boldsymbol{\alpha_{11}} e_1 + \boldsymbol{\alpha_{21}} e_2 + \cdots + \boldsymbol{\alpha_{n1}} e_n \\
e'_2 = \boldsymbol{\alpha_{12}} e_1 + \boldsymbol{\alpha_{22}} e_2 + \cdots + \boldsymbol{\alpha_{n2}} e_n \\
\vdots \\
e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n \\
\vdots \\
e'_n = \boldsymbol{\alpha_{1n}} e_1 + \boldsymbol{\alpha_{2n}} e_2 + \cdots + \boldsymbol{\alpha_{nn}} e_n
\end{gathered}
$$

Coordinate Change

$$x = x_1e_1 + x_2e_2 + \cdots + x_ne_n = x'_1\boldsymbol{e'_1} + x'_2\boldsymbol{e'_2} + \cdots x'_n\boldsymbol{e'_n} =$$

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Remember: $e_i' = \boldsymbol{\alpha_{1i}}e_1 + \boldsymbol{\alpha_{2i}}e_2 + \cdots + \boldsymbol{\alpha_{ni}}e_n$

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Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

Remember: $e'_i = \boldsymbol{\alpha_{1i}} e_1 + \boldsymbol{\alpha_{2i}} e_2 + \cdots + \boldsymbol{\alpha_{ni}} e_n$

$$= x'_1 \cdot (\boldsymbol{\alpha_{11} e_1} + \boldsymbol{\alpha_{21} e_2} + \cdots + \boldsymbol{\alpha_{n1} e_n}) + \cdots + x'_i \cdot (\boldsymbol{\alpha_{1i} e_1} + \boldsymbol{\alpha_{2i} e_2} + \cdots + \boldsymbol{\alpha_{ni} e_n}) + \cdots$$
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$e_1, \ldots, e_n$ linearly independent -> coefficients in front of them should be the same on the both sides of the equality:

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Coordinate Change

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = x'_1 \boldsymbol{e'_1} + x'_2 \boldsymbol{e'_2} + \cdots x'_n \boldsymbol{e'_n} =$$

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$$\boldsymbol{x_{old}} \quad \begin{array}{l} \boxed{x_1} = x'_1 \alpha_{11} + \cdots + x'_i \alpha_{1i} + \cdots + x'_n \alpha_{1n} \\ \boxed{x_2} = x'_1 \alpha_{21} + \cdots + x'_i \alpha_{2i} + \cdots + x'_n \alpha_{2n} \\ \qquad \vdots \\ \boxed{x_n} = x'_1 \alpha_{n1} + \cdots + x'_i \alpha_{ni} + \cdots + x'_n \alpha_{nn} \end{array}$$

Coordinate Change

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Coordinate Change

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 & & & & \vdots & & \\
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\end{array}$$

$\boldsymbol{x_{new}}$

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$

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- $x_{old} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix},\ \ x_{new} = \begin{bmatrix}x_1'\\x_2'\end{bmatrix}$

$$e_i'$$

$$x_{old} \quad \begin{matrix} x_1 = x_1'\alpha_{11} + \cdots + x_i'\alpha_{1i} + \cdots + x_n'\alpha_{1n} \\ x_2 = x_1'\alpha_{21} + \cdots + x_i'\alpha_{2i} + \cdots + x_n'\alpha_{2n} \\ \vdots \\ x_n = x_1'\alpha_{n1} + \cdots + x_i'\alpha_{ni} + \cdots + x_n'\alpha_{nn} \end{matrix}$$

$$x_{new}$$

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix}1\\0\end{bmatrix},\ e_2 = \begin{bmatrix}0\\1\end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix}2\\1\end{bmatrix},\ e'_2 = \begin{bmatrix}-1\\-1\end{bmatrix}$
- $x_{old} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix},\ \ x_{new} = \begin{bmatrix}x'_1\\x'_2\end{bmatrix}$

e'_i

$$x_{old}\qquad \begin{aligned} x_1 &= x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 &= x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ &\ \ \vdots \\ x_n &= x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{aligned}$$

x_{new}

$$\begin{aligned} 2 &= 2x'_1 - 1x'_2 \\ -1 &= 1x'_1 - 1x'_2 \end{aligned}$$

Coordinate Change: Example

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
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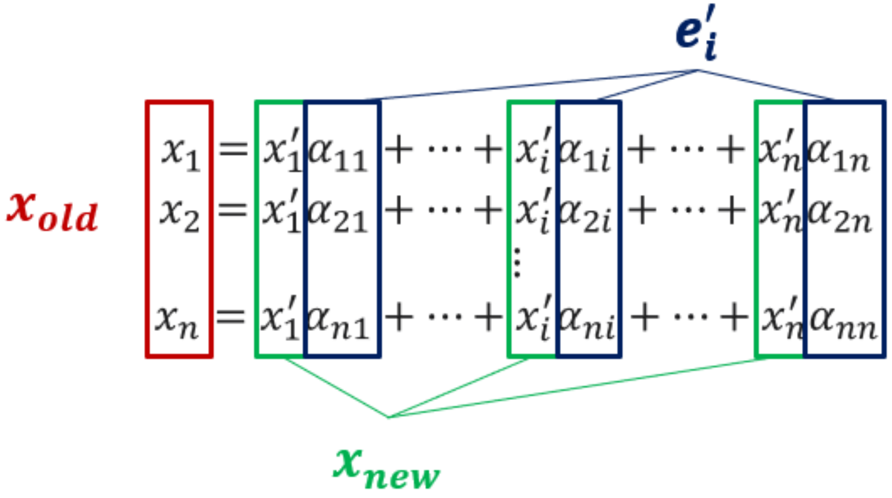
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \ \ x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$



$$\begin{matrix} 2 = 2x'_1 - 1x'_2 \\ -1 = 1x'_1 - 1x'_2 \end{matrix} \iff \begin{matrix} x'_1 = 3 \\ x'_2 = 4 \end{matrix}$$

Coordinate Change: Example

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- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$

$\boldsymbol{e'_i}$

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$\boldsymbol{x_{new}}$

$$\begin{aligned} 2 &= 2x'_1 - 1x'_2 \\ -1 &= 1x'_1 - 1x'_2 \end{aligned} \iff \begin{aligned} x'_1 &= 3 \\ x'_2 &= 4 \end{aligned} \iff x_{new} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Coordinate Change

- Going from one basis to the other:

$$e'_i$$

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Coordinate Change

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$$\boldsymbol{x_{new}}$$

- There is a more compact way of writing this down using matrices.

Matrices

A Matrix

- $A \in \mathbb{R}^{m \times n}$ - a matrix with m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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- *Examples:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \in \mathbb{R}^{2 \times 3}$$

Special Matrices

- Diagonal matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ii} \neq 0,\ a_{ij} = 0\ \forall i \neq j\right)$

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- Symmetric matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $\left(a_{ij} = a_{ji}\right)$
- Triangular matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ $\left(a_{ij} = 0\ \forall i > j\ or\ \forall i < j\right)$

Vectors vs Matrices

- An n-dimensional vector can be considered a $n \times 1$ matrix:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

Operations with Matrices

Transpose of a Matrix

- Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

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- Transpose = writing columns as rows:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T = [x_1, \dots, x_n]$$

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Transpose of a Matrix: Example

- $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 5 \end{bmatrix}$

Transpose of a Matrix: Example

- $\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 2 & 5 \end{bmatrix}$

- Transposing a symmetrical matrix = no changes:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}^{T} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Multiplying by a Scalar

- We can multiply matrix by a scalar:

$$\lambda A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{bmatrix}$$

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- Example:

$$5 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Sum of Two Matrices

- We can sum up matrices of the same size:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix},$$

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

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$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2n} + b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \cdots & a_{mn} + b_{mn} \end{bmatrix}$$

- Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 4 \end{bmatrix}$$

Matrices Also Form a Vector Space!

- $(\mathbb{R}^{m\times n}, +, \ \cdot)$ - a vector space.
  "Vectors" = matrices.

Matrices Also Form a Vector Space!

- $(\mathbb{R}^{m\times n}, +, \ \cdot)$ - a vector space.
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- You can check yourself that the necessary axioms hold.

Matrix Multiplication

- Consider two matrices $A = \{a_{ij}\}_{m\times n}$ and $b = \{b_{ij}\}_{n\times p}$.
- $C = AB$ – product of two matrices.

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- Example $\mathbb{R}^{2\times 2}$: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Matrix Multiplication: Example

$$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 6 & 4 \\ 2 & 5 & 8 \\ 7 & 1 & 9 \end{bmatrix} =$$

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$$= \begin{bmatrix} 0+2+14 & 0+5+2 & 0+8+18 \\ 6+2+35 & 12+5+5 & 8+8+45 \end{bmatrix} =$$

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$$= \begin{bmatrix} 0+2+14 & 0+5+2 & 0+8+18 \\ 6+2+35 & 12+5+5 & 8+8+45 \end{bmatrix} =$$

$$= \begin{bmatrix} 16 & 7 & 26 \\ 43 & 22 & 61 \end{bmatrix}$$

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Coordinate Change: Matrix Notation

- Result obtained before:

$e_1, \dots, e_n$ - old basis
$e'_1, \dots, e'_n$ - new basis

$x_{old} = [x_1, \dots, x_n], \qquad x_{new} = [x'_1, \dots, x'_n]$

e'_i

$$x_{old} \quad \begin{array}{l} x_1 = x'_1\alpha_{11} + \cdots + x'_i\alpha_{1i} + \cdots + x'_n\alpha_{1n} \\ x_2 = x'_1\alpha_{21} + \cdots + x'_i\alpha_{2i} + \cdots + x'_n\alpha_{2n} \\ \qquad\qquad\qquad \vdots \\ x_n = x'_1\alpha_{n1} + \cdots + x'_i\alpha_{ni} + \cdots + x'_n\alpha_{nn} \end{array}$$

x_{new}

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$$\boldsymbol{x_{new}}$$

- Transition matrix: columns = coordinates of the new basis in the old one.

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{21} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}$$

Coordinate Change: Matrix Notation

- Result obtained before:

  $e_1, \dots, e_n$ - old basis
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$$e'_i$$

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$$x_{old} = A^T x_{new}$$

Coordinate Change: Example (again)

- Consider $\mathbb{R}^2$ with basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- New basis: $e'_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ e'_2 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$
- $x_{old} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},\ x_{new} = \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = ?$

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- $x_{old} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}2\\-1\end{bmatrix}$, $x_{new} = \begin{bmatrix}x'_1\\x'_2\end{bmatrix} = ?$

$$A = \begin{bmatrix}2 & -1\\1 & -1\end{bmatrix},$$

Coordinate Change: Example (again)

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$$A = \begin{bmatrix}2 & -1\\1 & -1\end{bmatrix}, \qquad \begin{bmatrix}2\\-1\end{bmatrix} = x_{old} = A^T x_{new} = \begin{bmatrix}2 & 1\\-1 & -1\end{bmatrix}\begin{bmatrix}x'_1\\x'_2\end{bmatrix} = \begin{bmatrix}2x'_1 + x'_2\\-x'_1 - x'_2\end{bmatrix}$$

$$x_{new} = \begin{bmatrix}3\\4\end{bmatrix}.$$

To Sum Up

- Vector spaces
  - Linear (in)dependence
  - Span
  - Basis

- Matrices
  - Matrix operations
  - Change of coordinates

Next Time

- More on matrices
- Systems of linear equations

Math Refresher for DS

Lecture 3

Last Time

- Vector Spaces
    - Linear combinations
    - Spans
    - Bases
    - Change of coordinates
- Matrices

Today

- More on matrices
  - matrix operations;
  - rank;
  - determinant.
- Linear transformations
- Systems of linear equations

Matrices: a small review

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- Identity matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\left(a_{ii} = 1,\ a_{ij} = 0\ \forall i \neq j\right)$

- Symmetric matrix: $\begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}$ $\left(a_{ij} = a_{ji}\right)$

- Triangular matrix: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ $\left(a_{ij} = 0\ \forall i > j\ or\ \forall i < j\right)$

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Basic Operations with Matrices

- Addition:

$$A = \{a_{ij}\}_{i=1,\ldots,m,j=1,\ldots,n}, \qquad B = \{b_{ij}\}_{i=1,\ldots,m,j=1,\ldots,n}, \qquad A + B = \{a_{ij} + b_{ij}\}_{i=1,\ldots,m,j=1,\ldots,n}$$

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- Multiplication by a scalar:

$$A = \{a_{ij}\}_{i=1,\dots,m,j=1,\dots,n}, \qquad \lambda \in \mathbb{R}, \qquad \lambda A = \{\lambda a_{ij}\}_{i=1,\dots,m,j=1,\dots,n}$$

Matrix Multiplication

- Matrix multiplication:

$$A = \{a_{ij}\}_{i=1,\ldots,m,j=1,\ldots,n}, \qquad B = \{b_{ij}\}_{i=1,\ldots,n,j=1,\ldots,k}$$

$$A \cdot B = \{(A_i, B^j)\}_{i=1,\ldots,m,j=1,\ldots,k} = \left\{\sum\nolimits_{l=1,\ldots,n} a_{il} \cdot b_{lj}\right\}_{i=1,\ldots,m,j=1,\ldots,k}$$

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- Example $\mathbb{R}^{2\times 2}$:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

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- Example:

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & 2 \\ 3 & 2 \end{bmatrix}, \quad AB = \begin{bmatrix} 30 & 14 \\ 12 & 6 \end{bmatrix}, \quad BA = \begin{bmatrix} 20 & 28 \\ 11 & 16 \end{bmatrix}$$

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Matrix Multiplication

- Multiplication by identity matrix E:

$$AE = EA = A$$

Matrix Multiplication

- Multiplication by identity matrix E:

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- Multiplication by zero matrix O:

$$AO = OA = O$$

Transposing a Matrix

- The transpose of a matrix results from “flipping” the rows and columns:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

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Transposing a Matrix

- The following properties of transposes are easily verified:
  - A – symmetric matrix $\Longrightarrow A^T = A$
  - $(A^T)^T = A$
  - $(A+B)^T = A^T + B^T$
  - $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$

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Linear Transforms

A more interesting way of looking at matrices.

Linear Transformation

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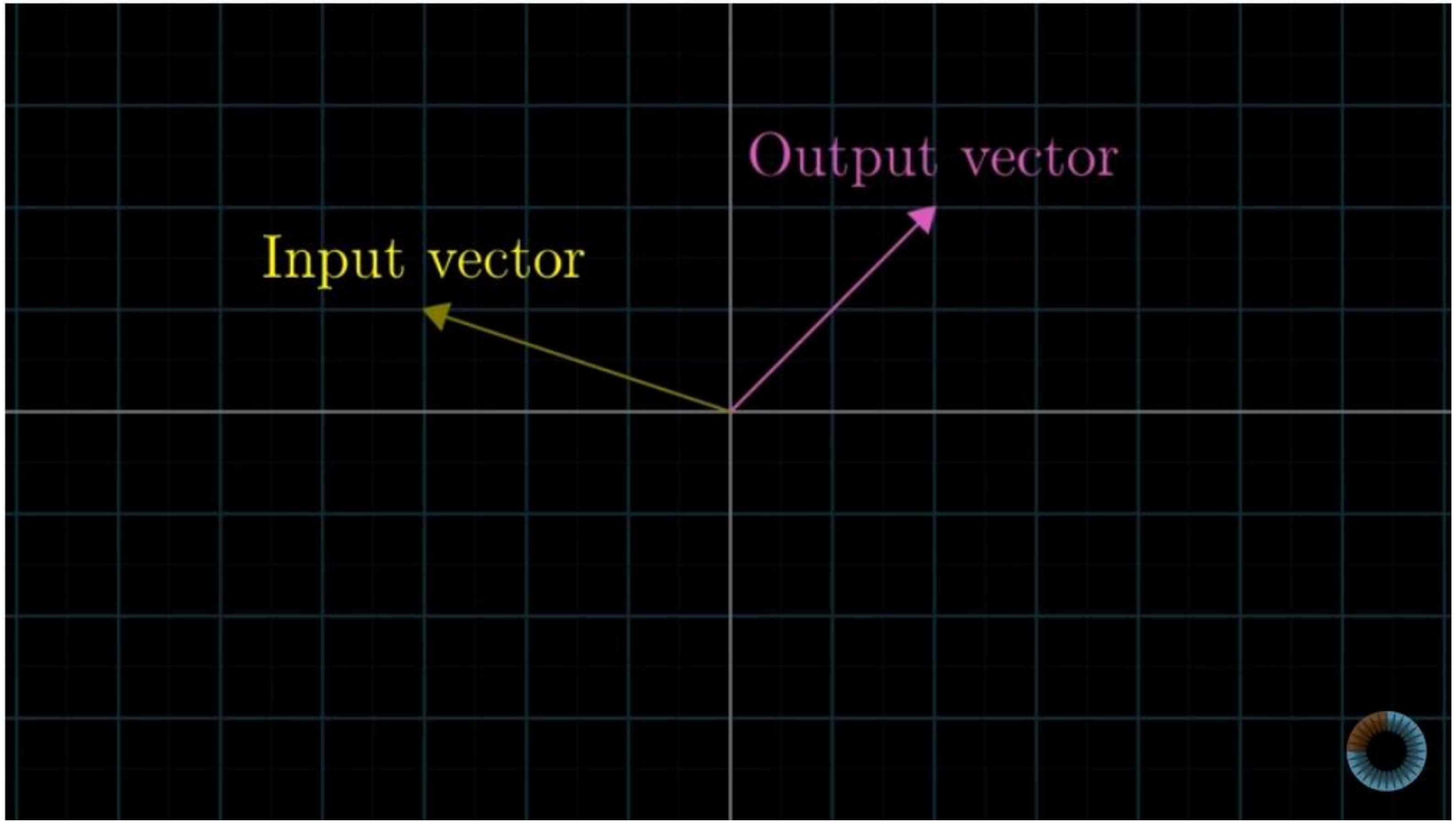
Linear Transformation

$$x_{input} \to A \to x_{output}$$

A – transformation

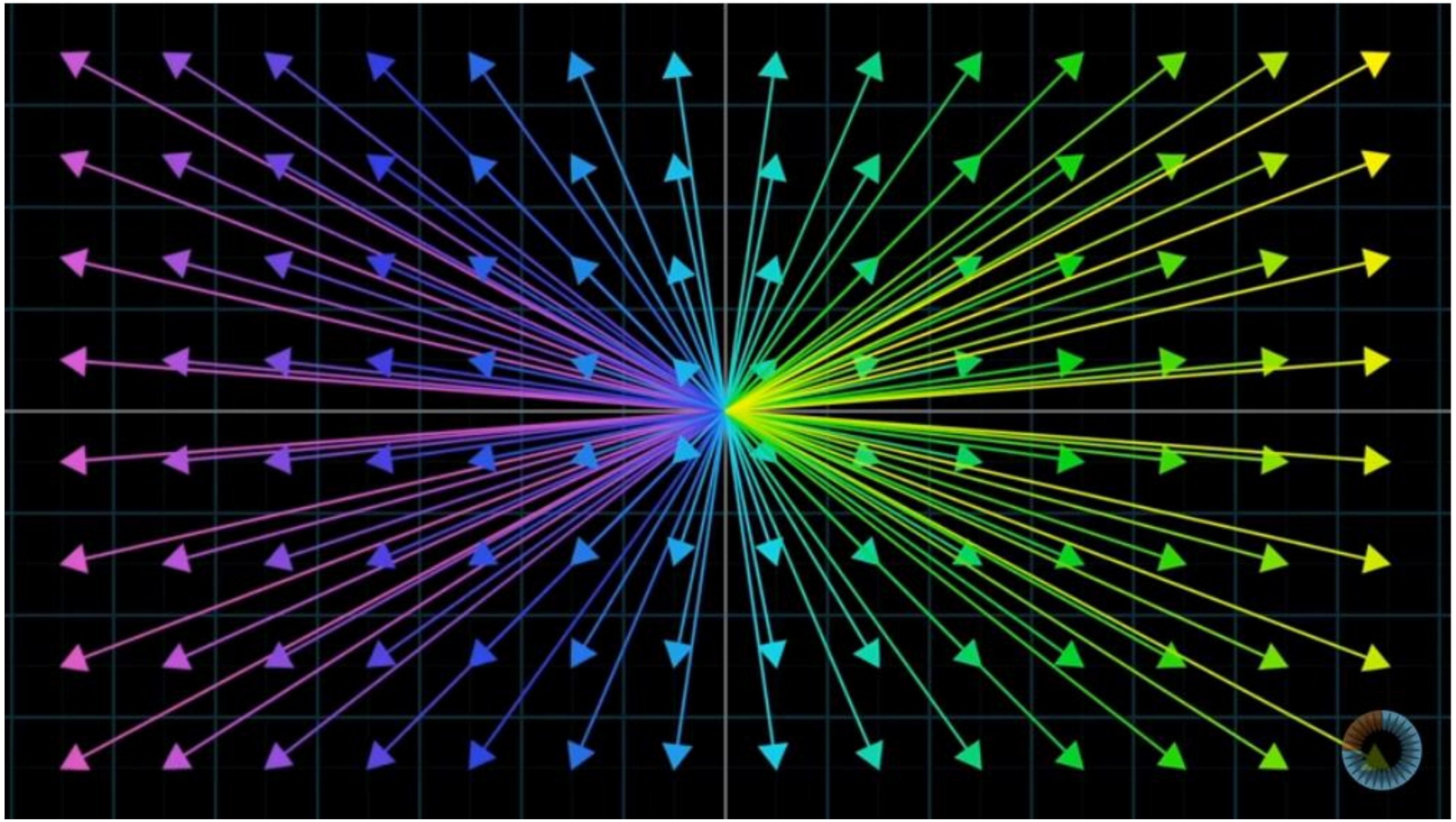
x_{input}, x_{output} – vectors

Transformation



Source: [Linear Transformations and Matrices]

Transformation



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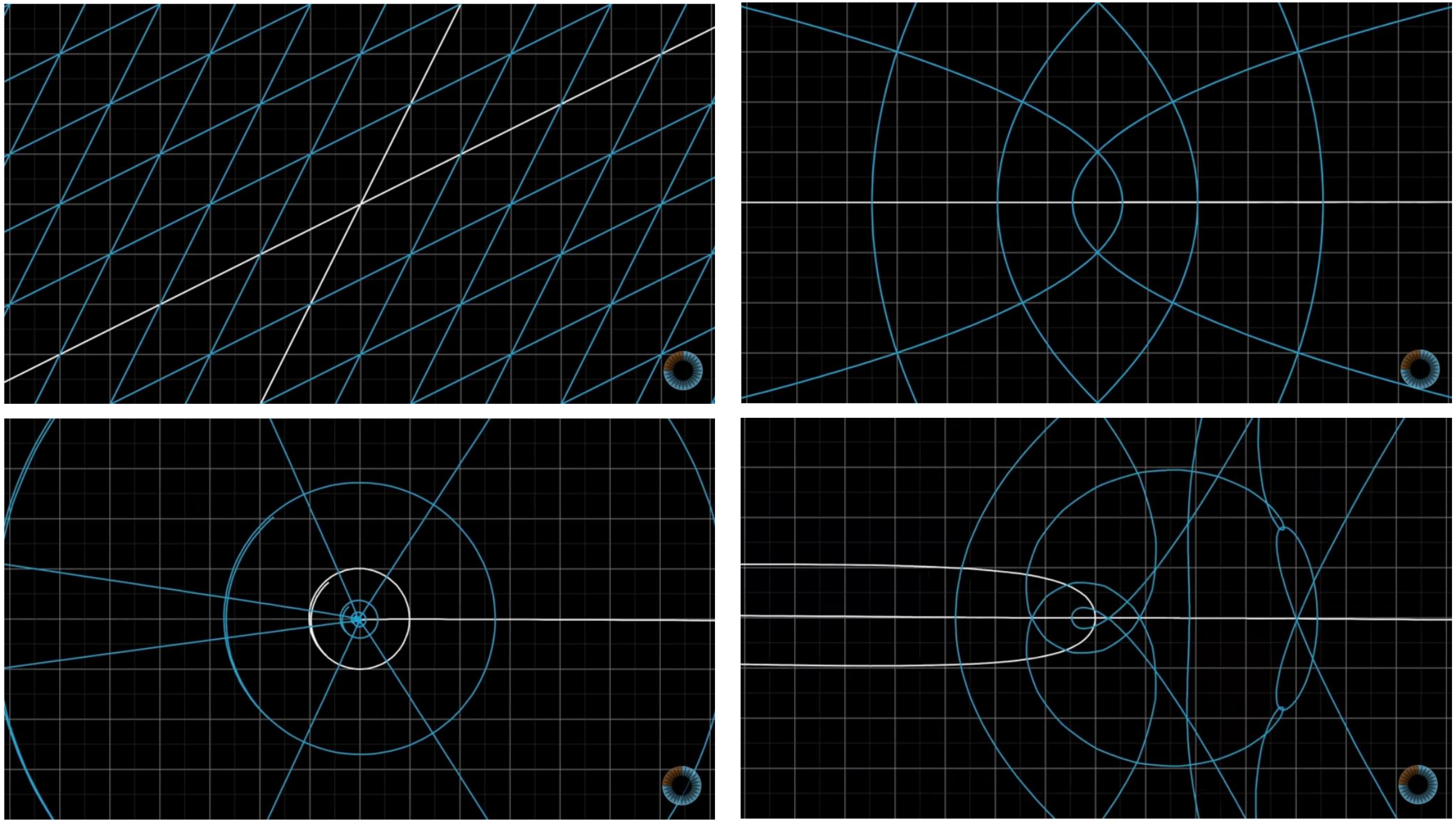
Transformation: Examples



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Linear Transformation

<u>Linear</u> Transformation

$$x_{input} \rightarrow A \rightarrow x_{output}$$

A – transformation

x_{input}, x_{output} – vectors

Linear Transformation

<u>Linear</u> Transformation

A transformation that satisfies two properties:

1. $A(x + y) = A(x) + A(y)$

2. $A(\lambda x) = \lambda A x$

$$x_{input} \to A \to x_{output}$$

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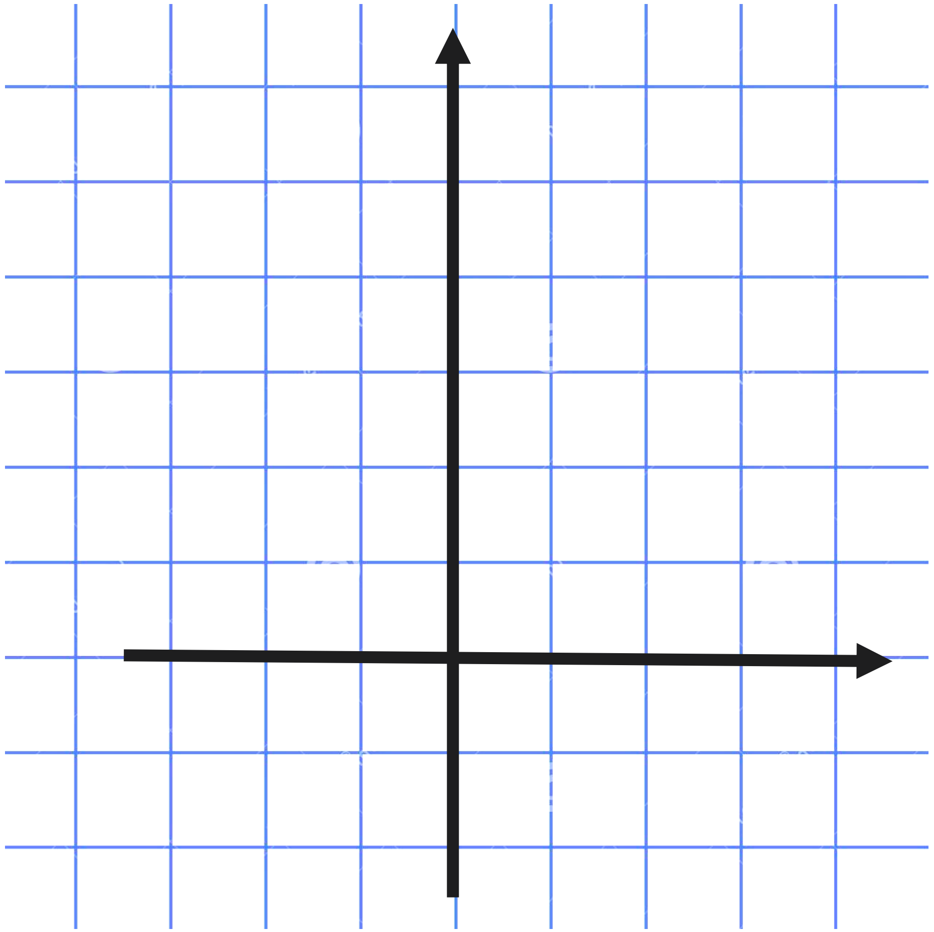
Example: Rotation

- Imagine that we want to rotate vectors in $\mathbb{R}^2$ 90° anti-clockwise.



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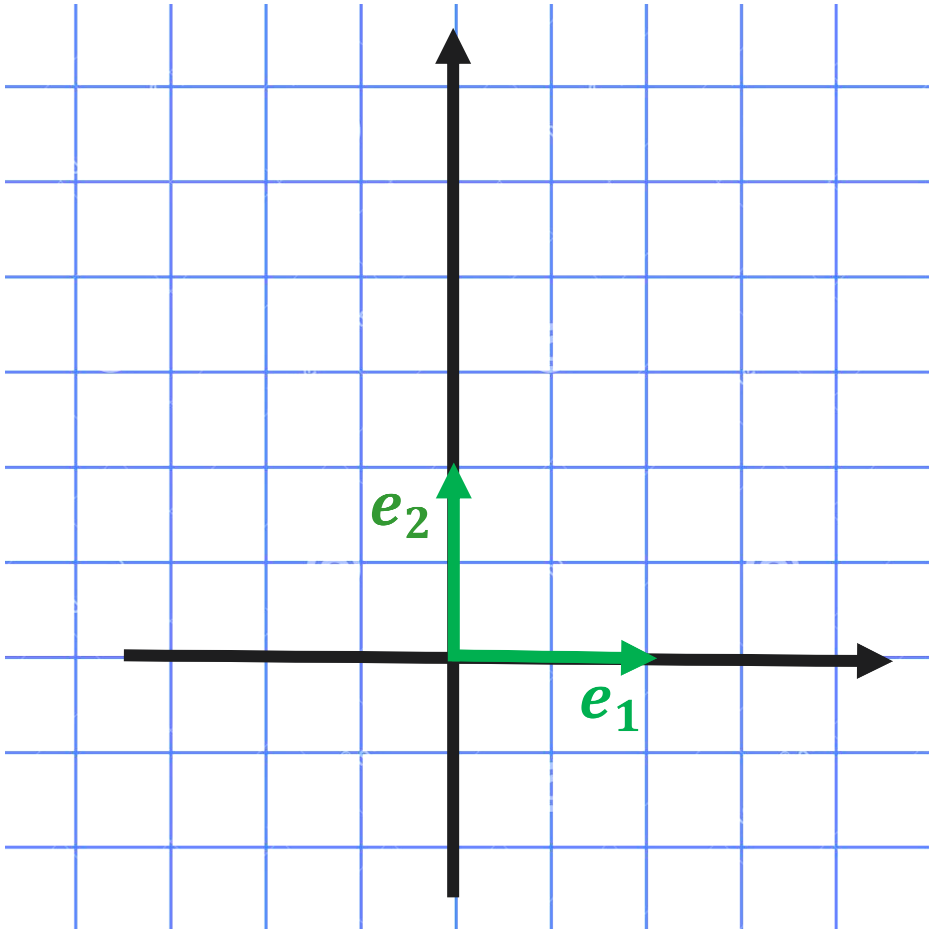

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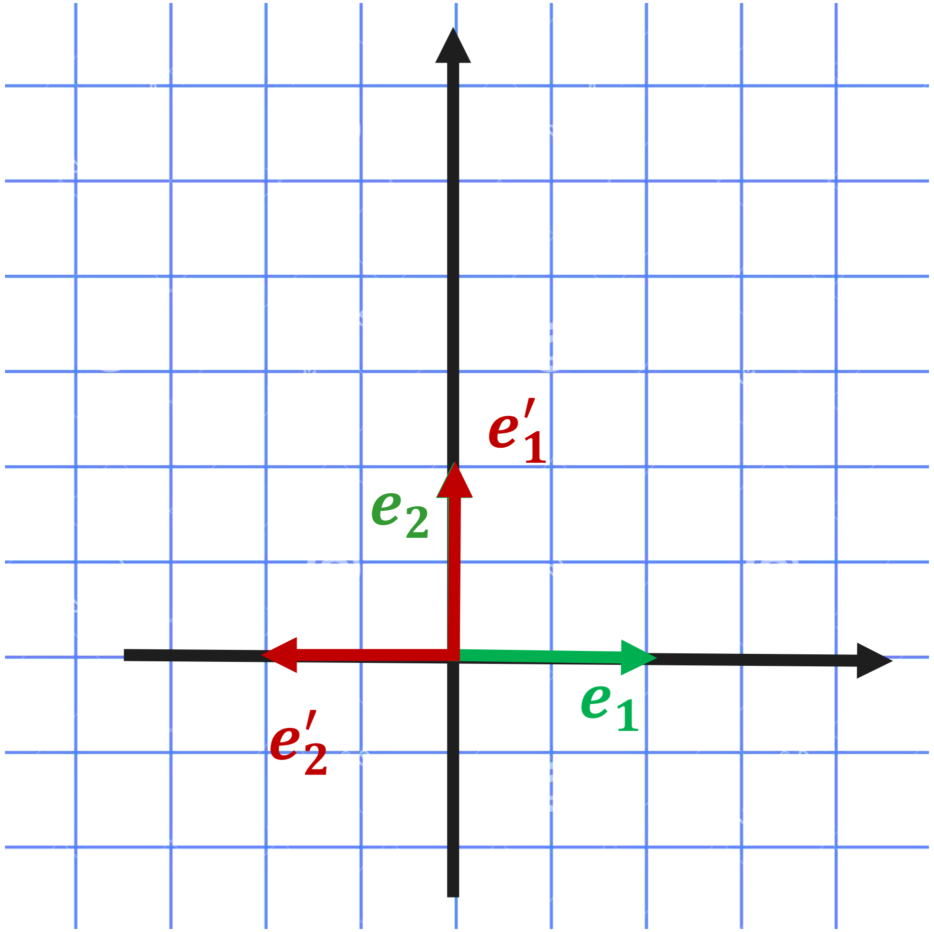

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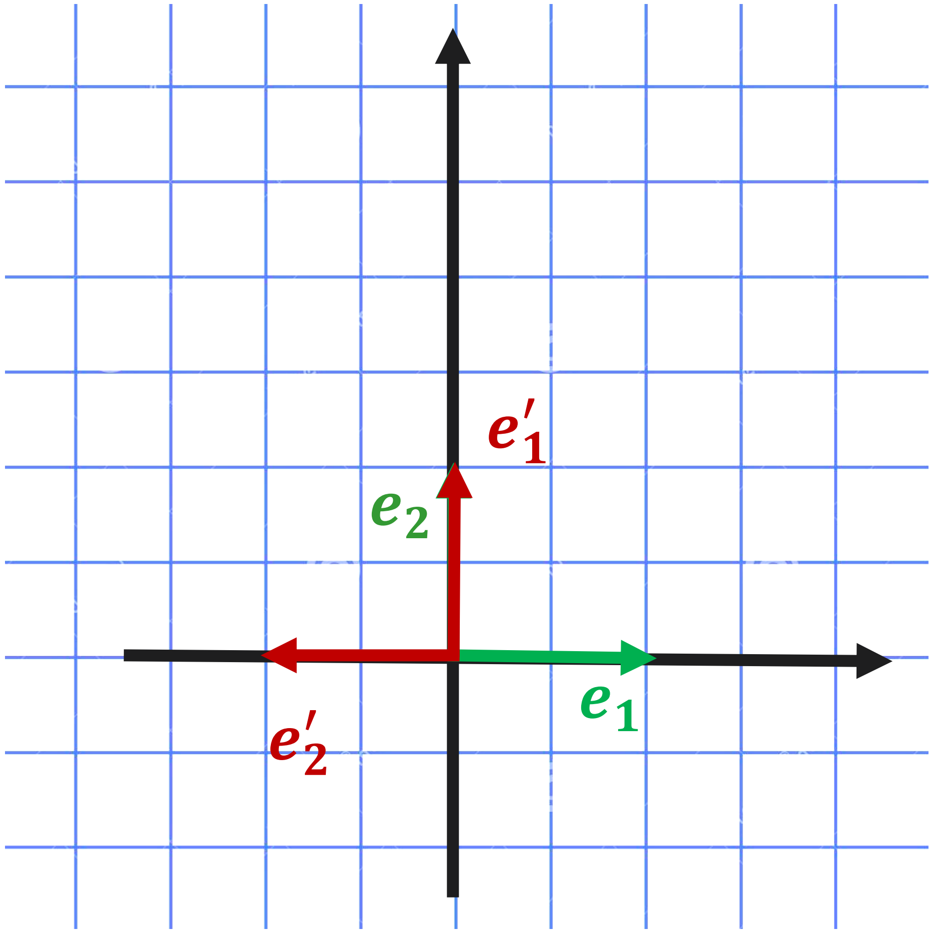

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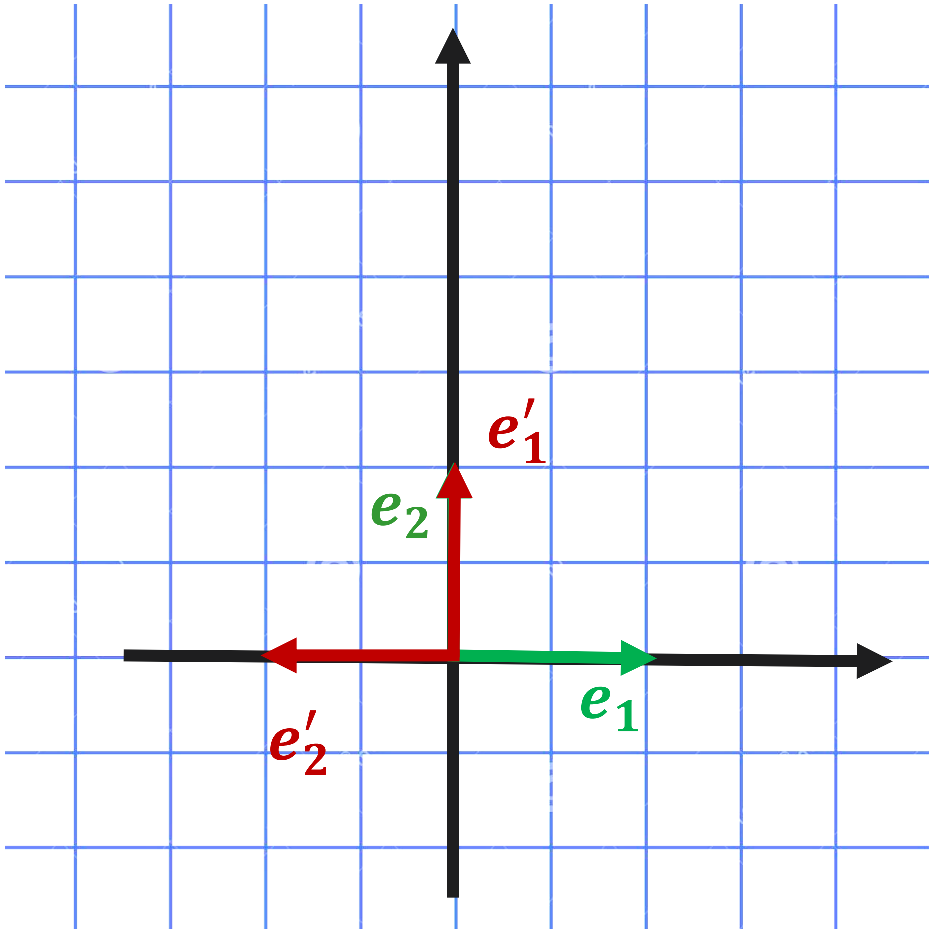

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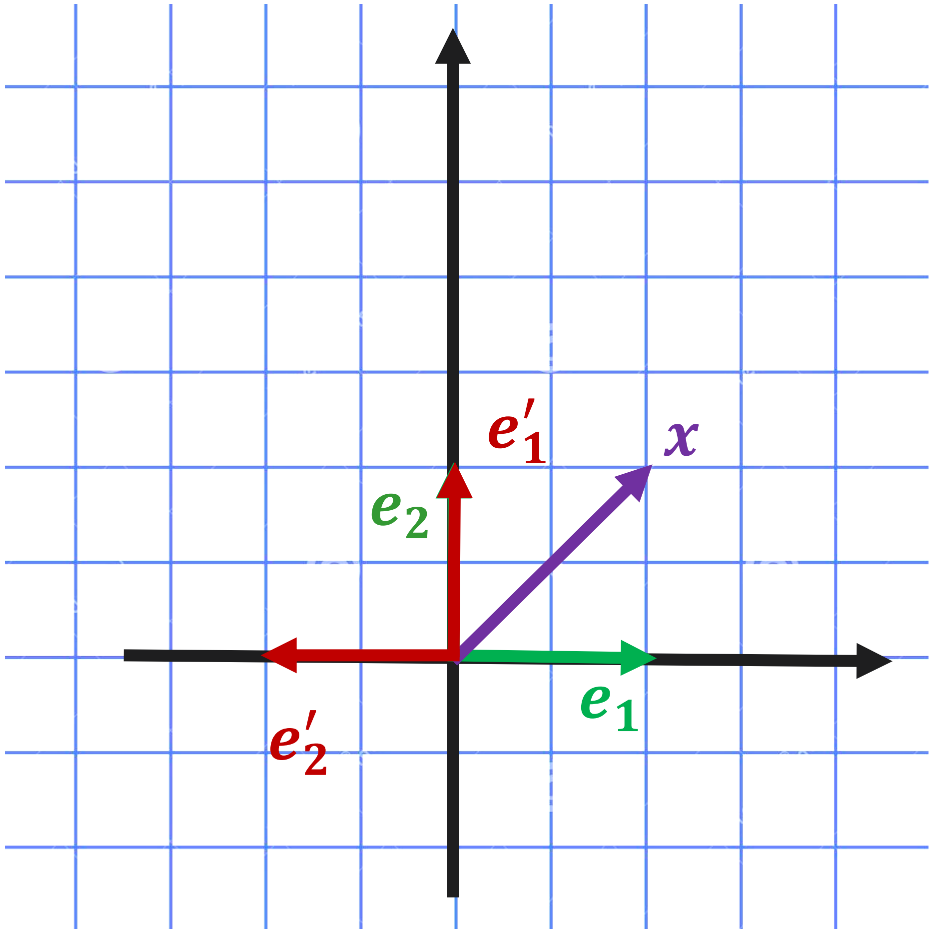

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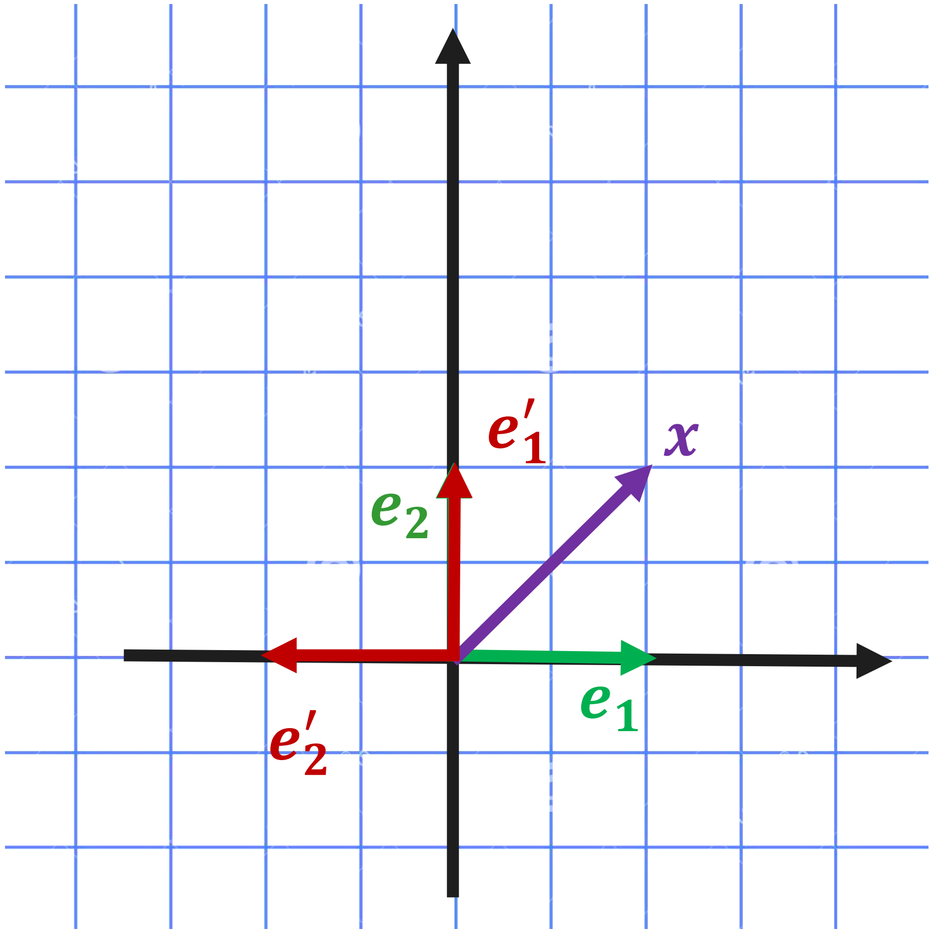

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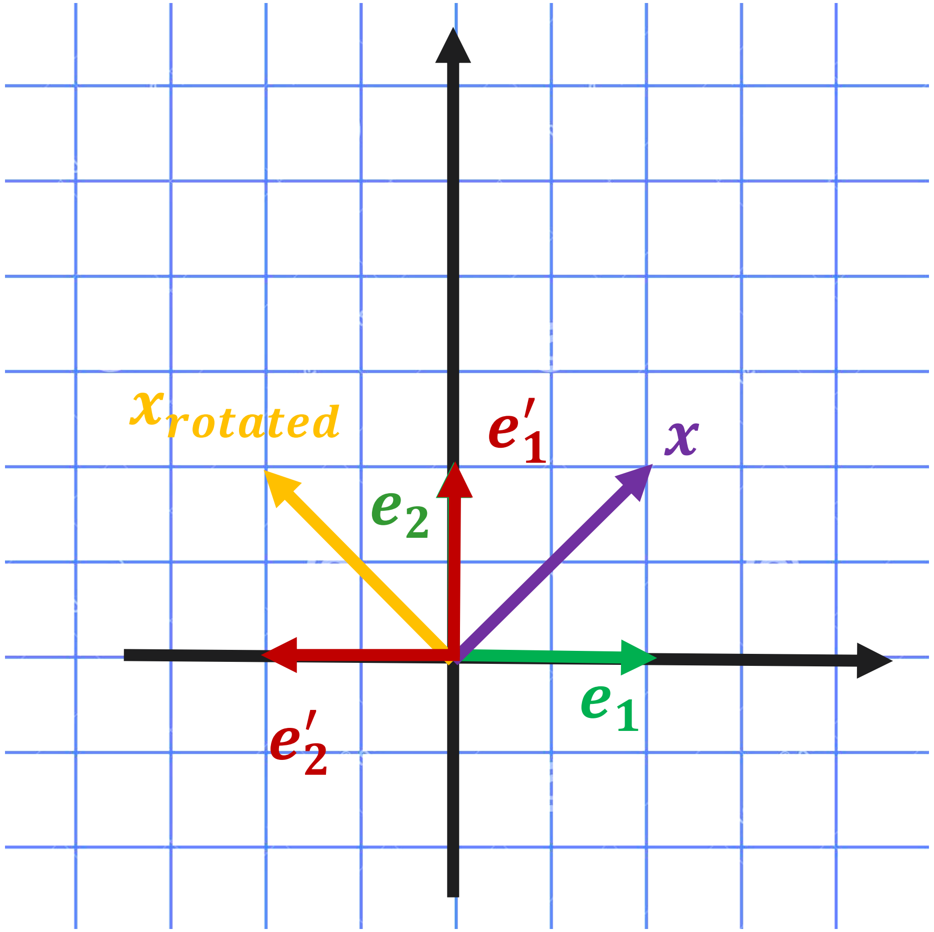

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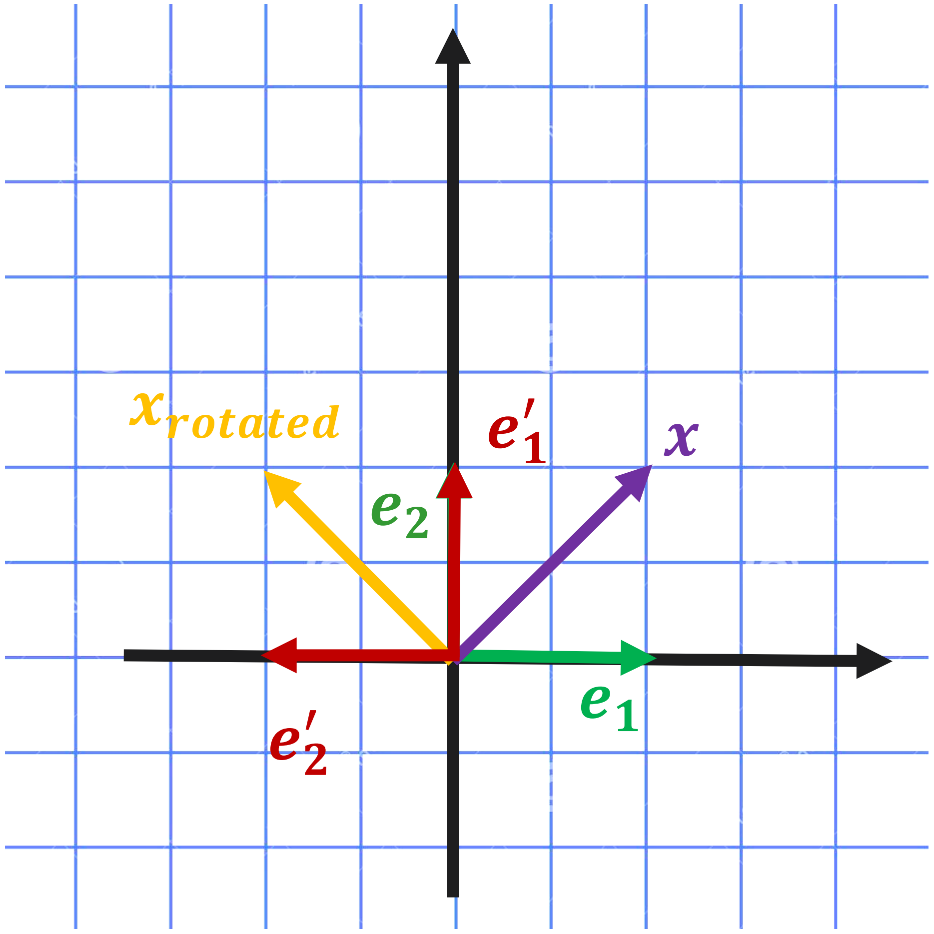

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- Vice versa: every square matrix defines some linear transformation.

Common Transforms

Identity Transformation

- Doesn't change anything.

- Transformation matrix E:

$$Ex = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Stretching / Squeezing

- Enlarge (compress) all distances in a particular direction by a constant factor.

- Transformation matrix:

$$Kx = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- Example: stretch x-axis (x3) and squeeze y-axis (x 0.5):

$$\begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Projection on an Axis

- Consider $\mathbb{R}^3$. Project on the XY −plane.

- Transformation matrix:

$$Px = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

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Rotation

- Rotating points anticlockwise by θ.
- Rotation matrix R_θ:

$$R_\theta\, x = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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- <u>Example:</u> rotate by 45° anticlockwise:

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}$$

Combining Transforms

Composition

- Let A and B be two linear transforms.
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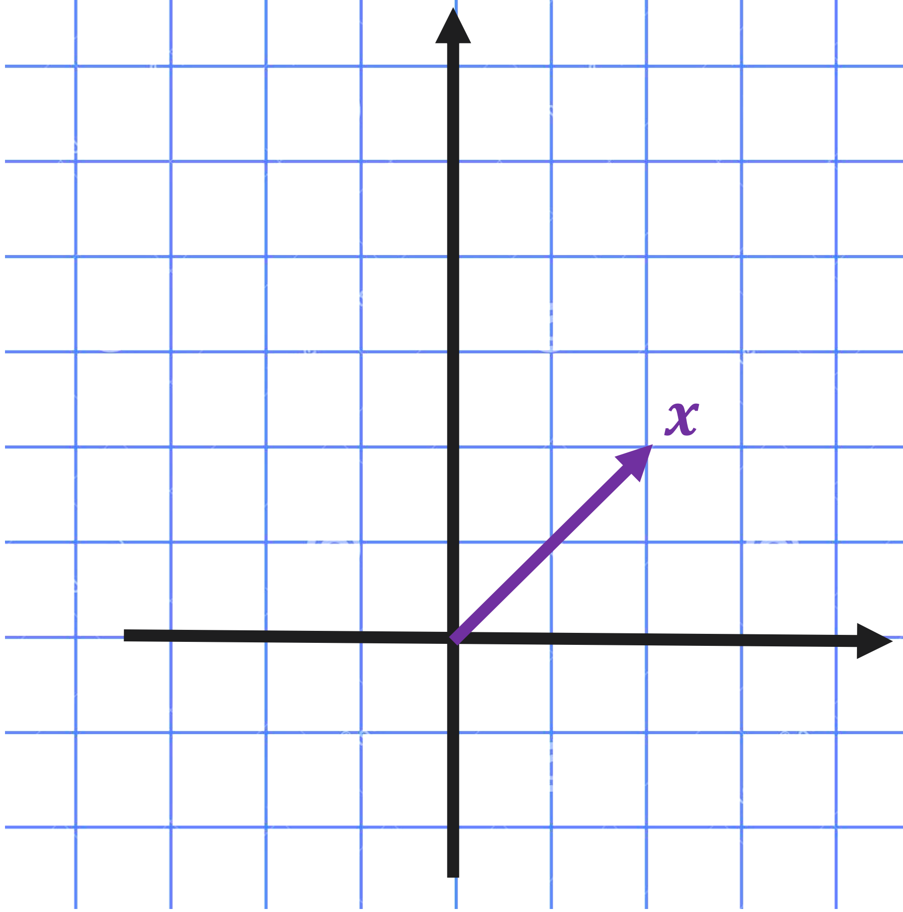

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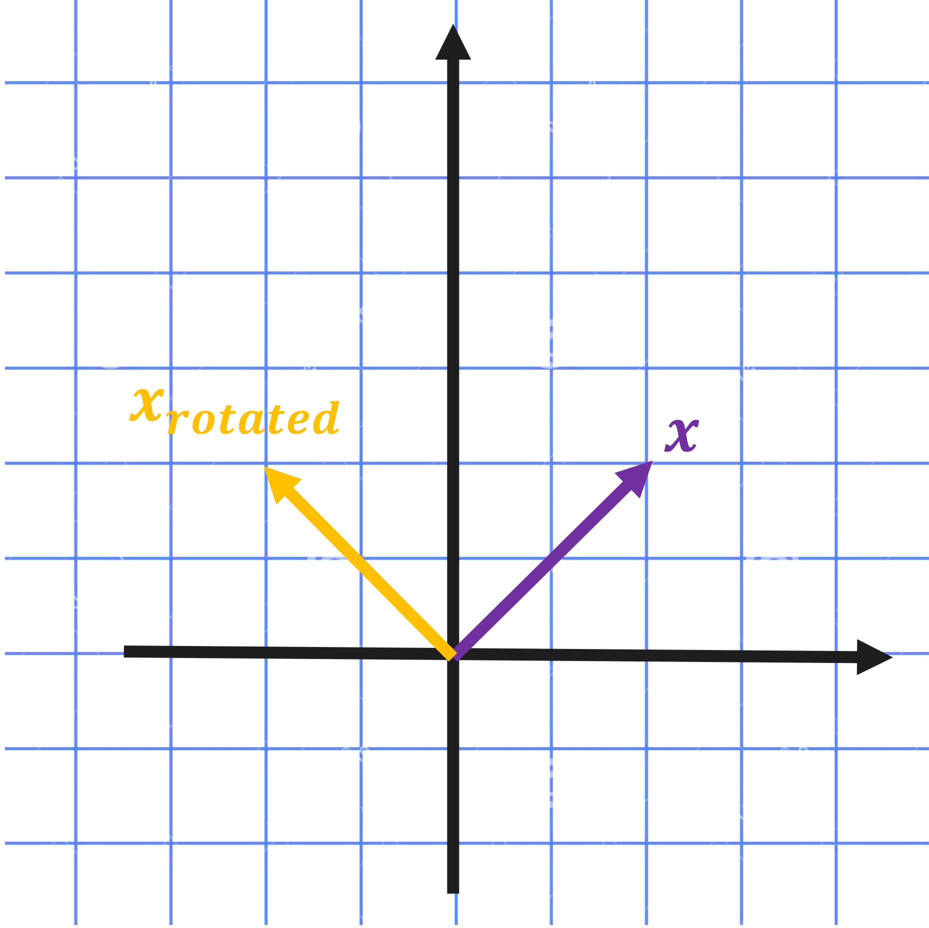

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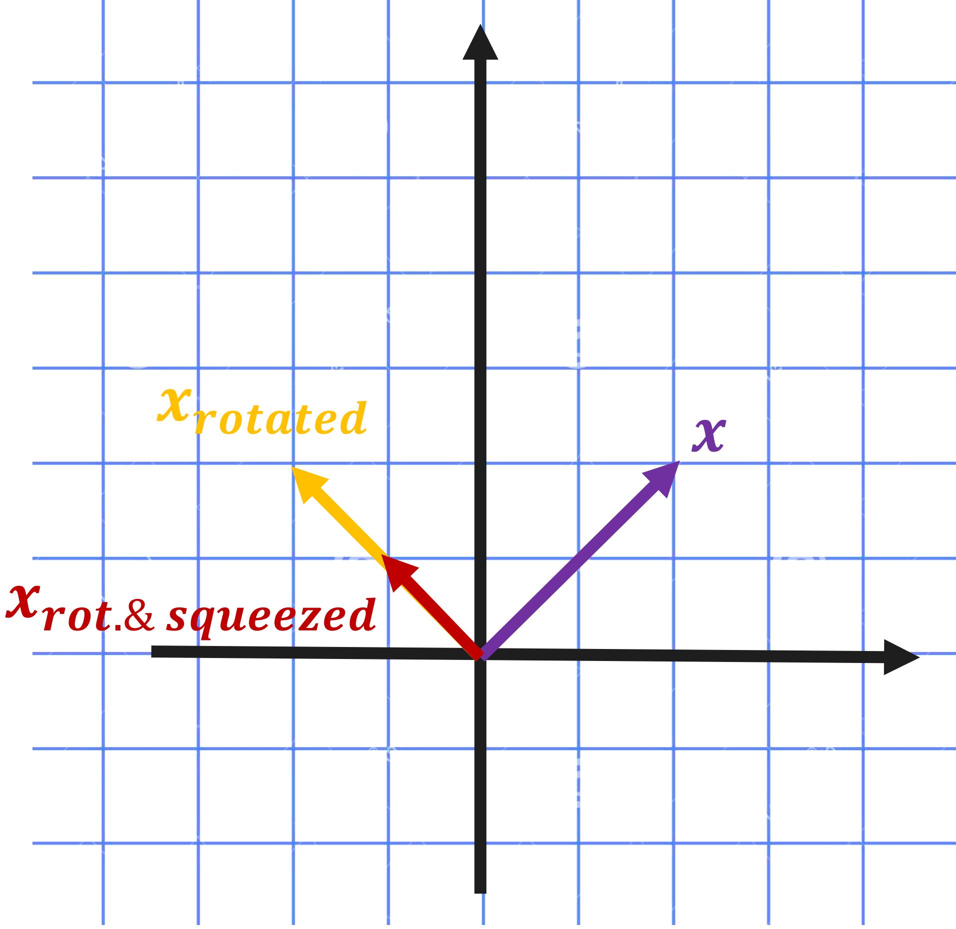

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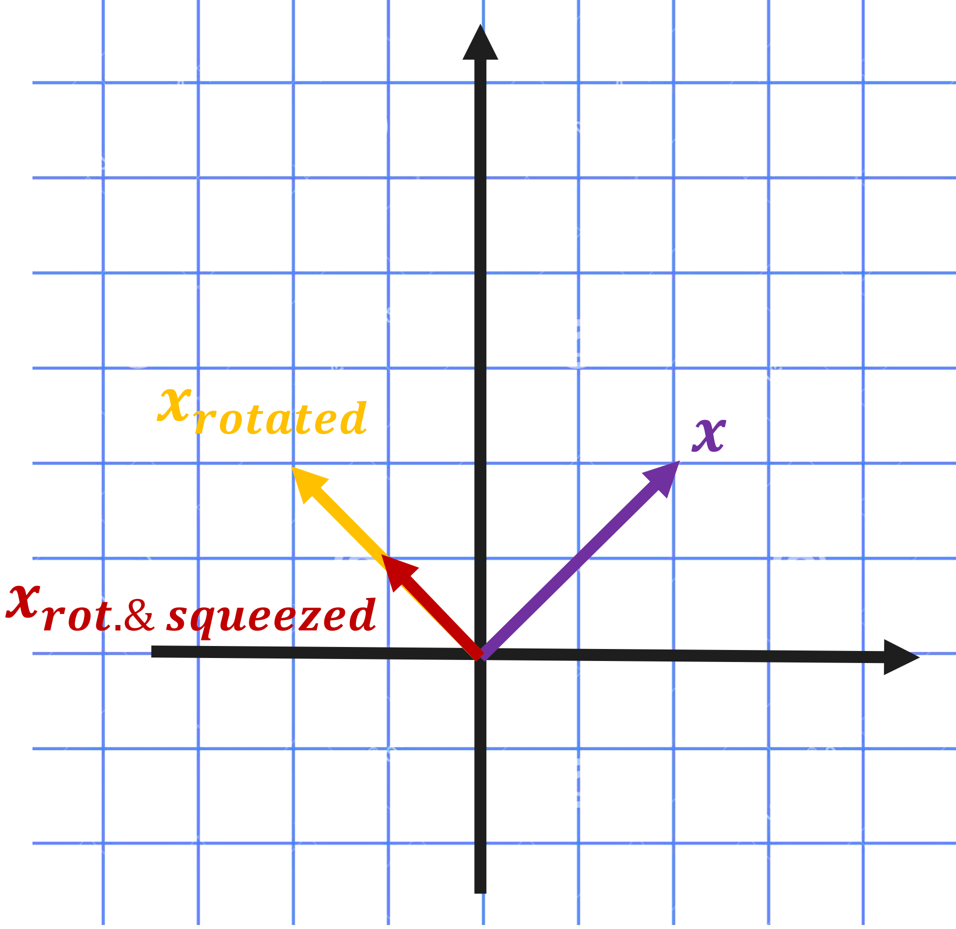

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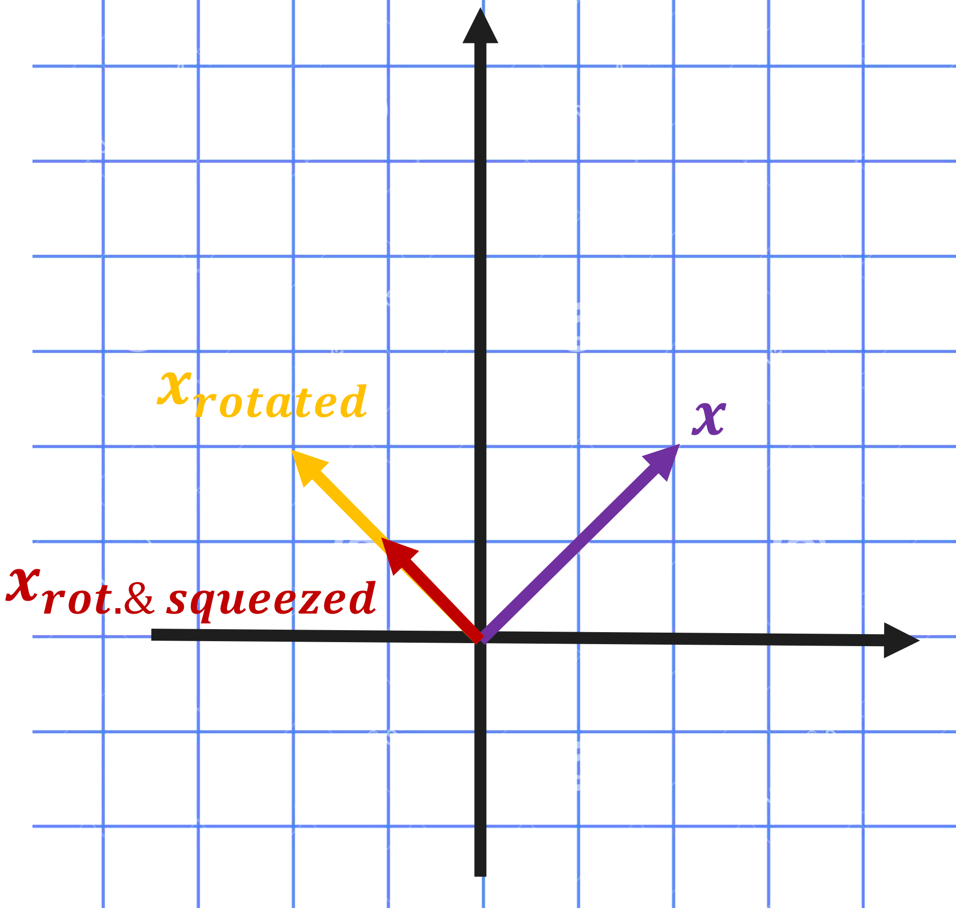

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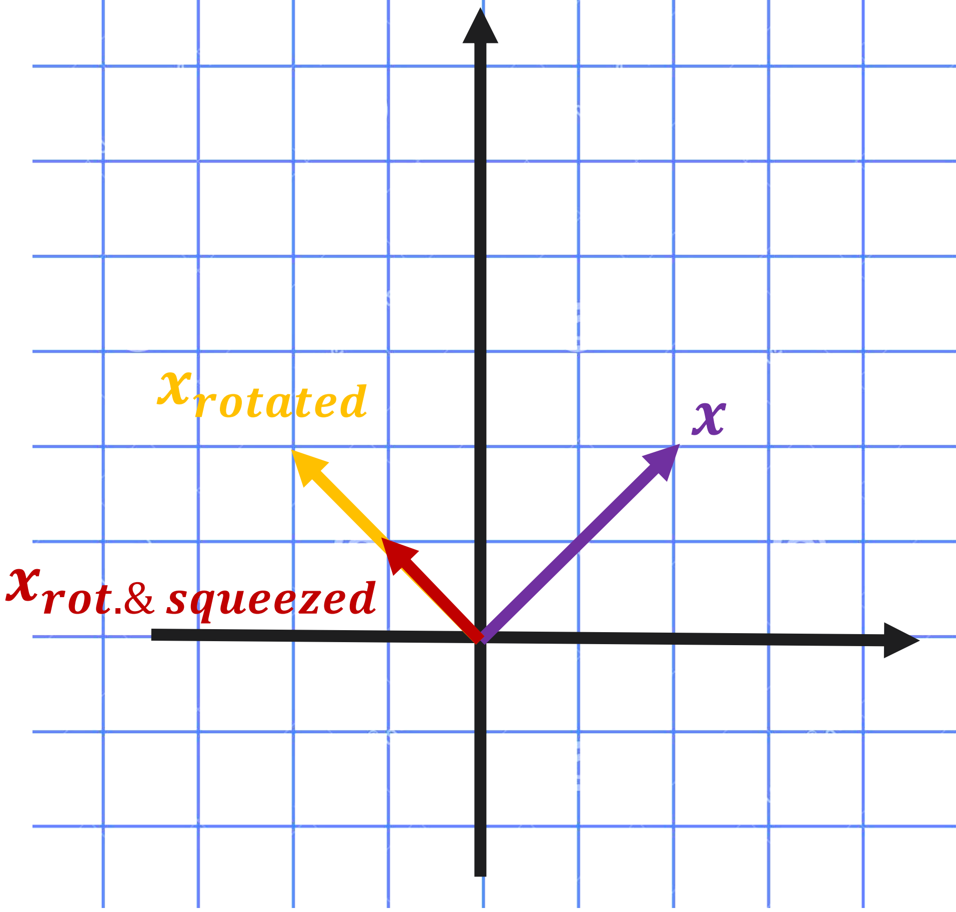

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= "rotate by 90° and squeeze"

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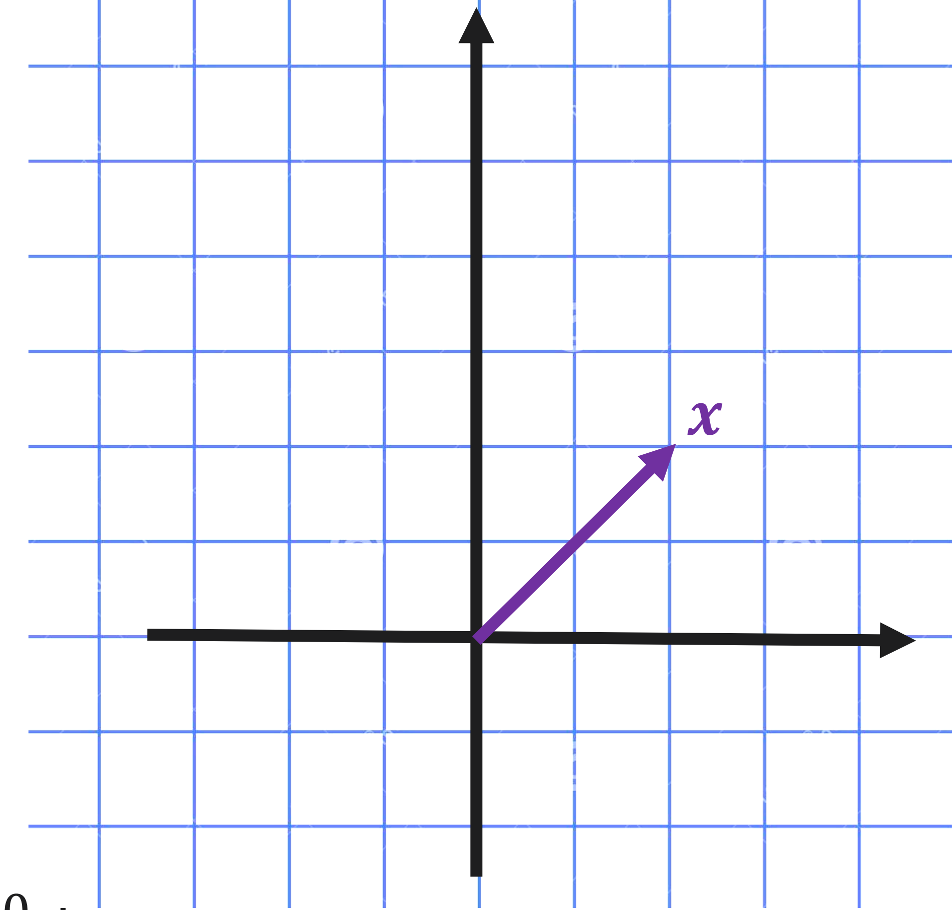

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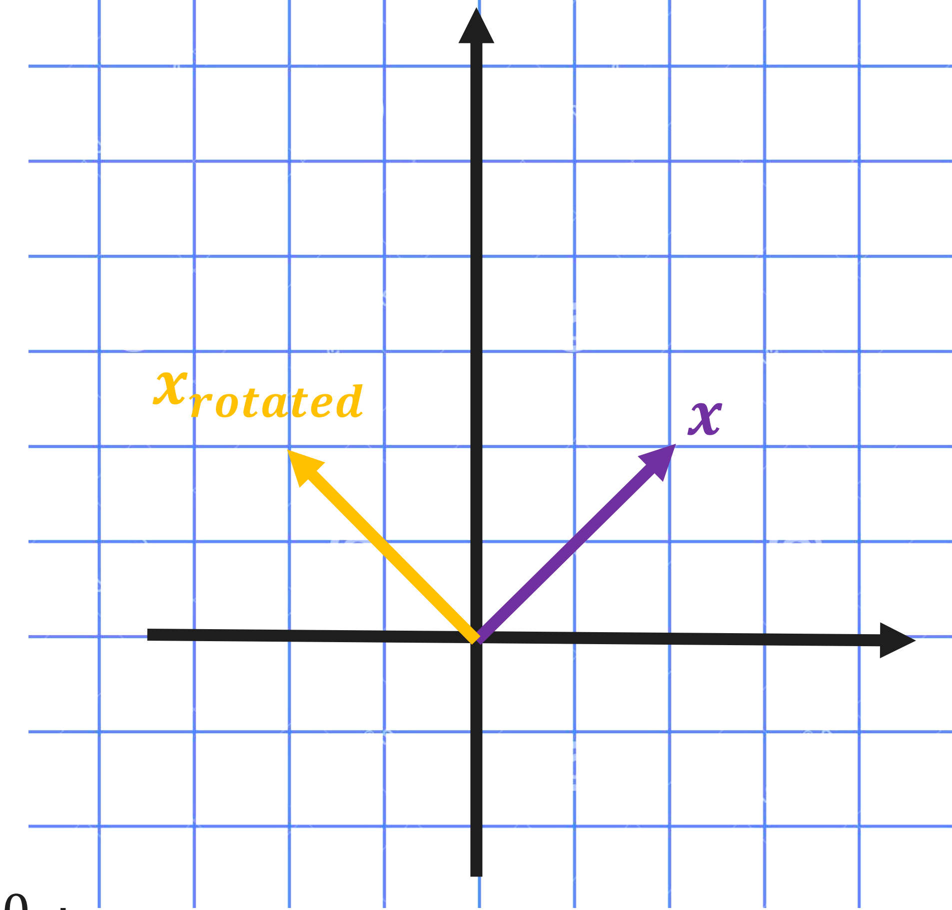

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \text{rotation}, \quad B = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} - \text{squeezing}$$

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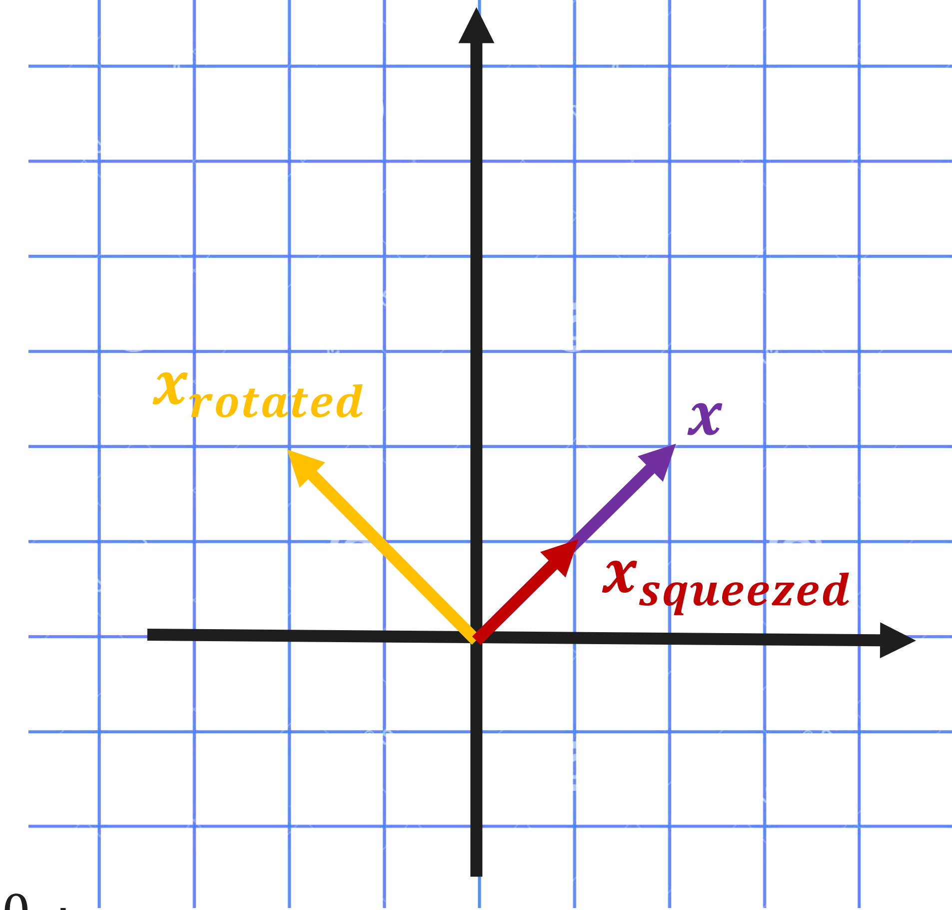

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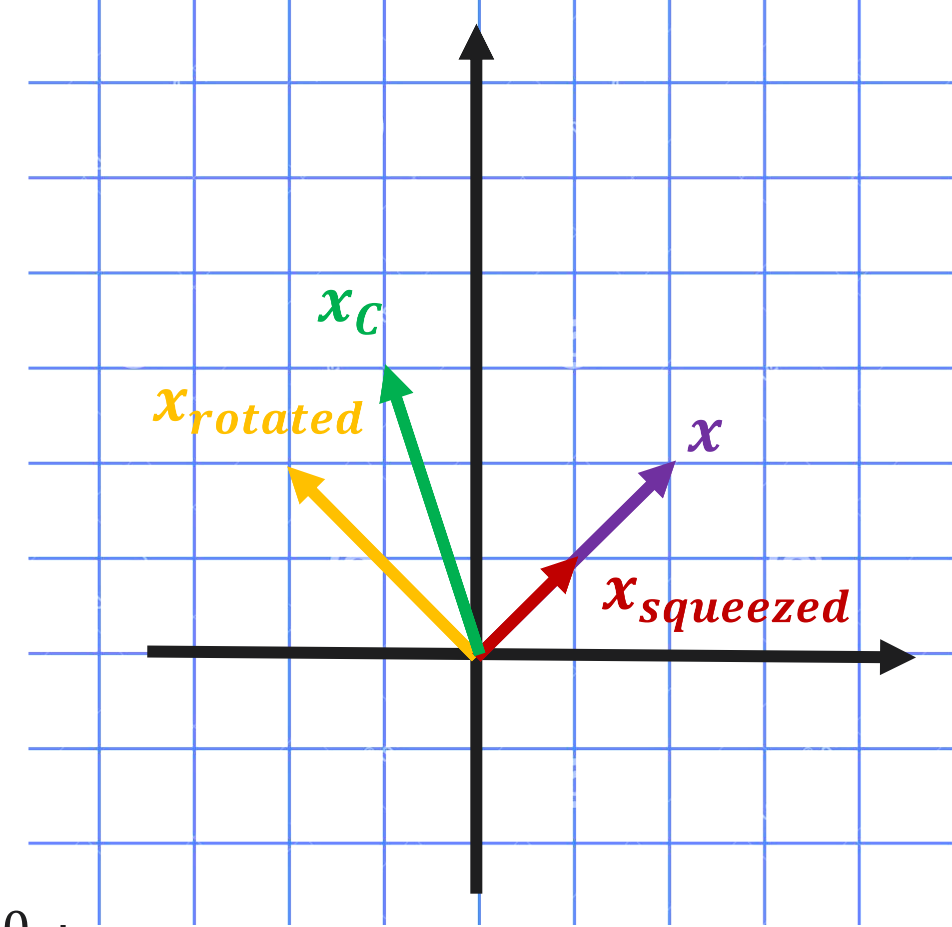

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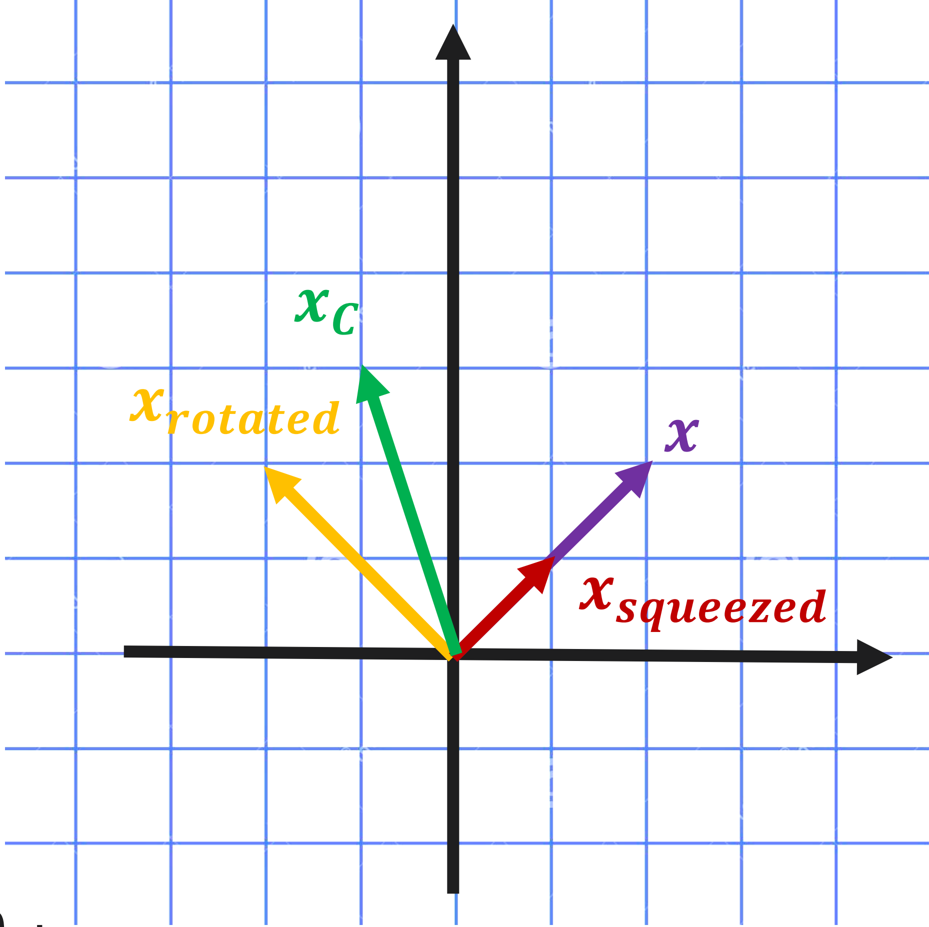

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- Which matrices have an inverse?

Determinant

Determinant

- A numerical way to characterize a linear transformation (and its matrix):
    - absolute value = how much area changes;
    - sign = change of orientation.

- More info on the interpretation: see video.

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Determinant

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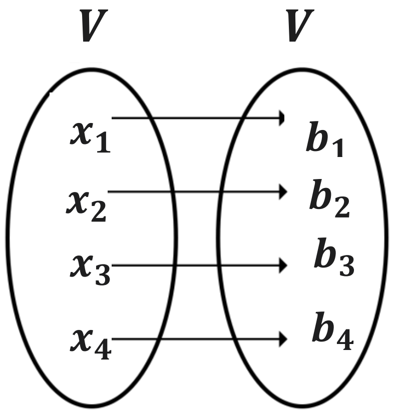
  - $\det A \neq 0$:
    - A:



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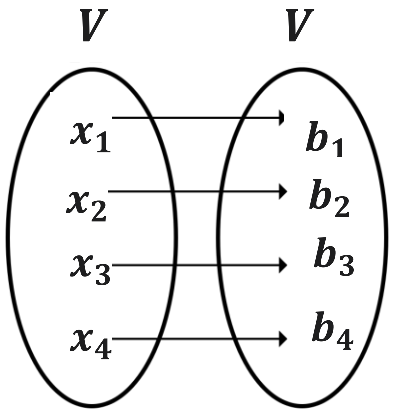

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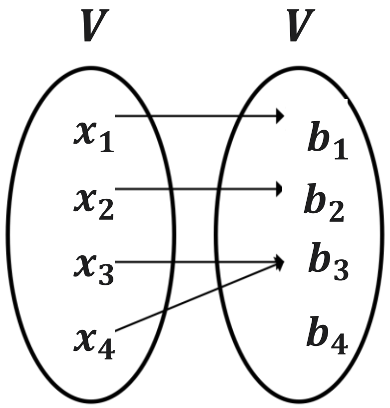

Computing Determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

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- Example:

$$R_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0 - (-1) = 1 \iff$$

“there is a transform inverse to rotation by $\mathbf{90}^{\circ}$ anticlockwise”.

Computing Determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

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Computing Determinant

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- Example:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 0 + 0 = 0 \Leftrightarrow$$

"there is no transpose inverse to projection onto XY-plane"

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- $A = \{a_{ij}\} - n \times n$ matrix.

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- Laplace extension.

Some Properties of the Determinant

- $\det A^T = \det A$
- $\det AB = \det A \cdot \det B$
- $\det A^{-1} = \dfrac{1}{\det A}$

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Finding Inverse of a Matrix

Gaussian Elimination

- $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

- $\det A \neq 0 \Longrightarrow$ there exists A^{-1}. Let's find it!

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- $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
- $\det A \neq 0 \Rightarrow$ there exists A^{-1}. Let's find it!
- Augment the initial matrix:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

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- Perform **elementary row operations** and obtain identity matrix on the left. The inverse will be on the right!

Gaussian Elimination

- Elementary row operations:
  - swap rows;
  - multiply rows by some number;
  - add / subtract one row to / from another.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

Gaussian Elimination

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \{(3) - (1)\} \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \to$$

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$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix}, \qquad AA^{-1} = A^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rank

Column Space

- Consider a square matrix A.
- Its columns $A^1, \dots, A^n$ can be seen as vectors.

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- $U = span\{A^1, \dots, A^n\}$ – column space of A.
  - All vectors that can be obtain by linearly combining columns of A.
  - $\Leftrightarrow$ image of linear transformation A (= all the vectors we can get by applying A).

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- Column space $U = span\{A^1, \ldots, A^n\}$ is the image of linear transformation A.

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- Rank of a matrix is the number of dimensions in its column space.
    - Full rank matrix: n columns, all linearly independent.
    - Lower-rank matrices: linearly dependent columns present.

Rank: Examples

- $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

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Column vs Row Rank

- Column space of A = span of A's columns.
  Its dimensionality = (column) rank.

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- Fundamental result: the column rank and the row rank are always equal.
  See proofs.

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$$rank(X) \leq \min\{n, m\}$$

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  Infinitely many vectors are mapped into a zero vector.

- $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \ rank(A) = 2 \Leftrightarrow \mathbb{R}^3$is mapped onto a plane

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- $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ rank(A) = 3 \Leftrightarrow \mathbb{R}^3$is mapped on itself (isomorphism)

  Only a zero vector is mapped into a zero vector.

Null space

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- Example: projection onto XY-plane:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow$$

Null space: $\left\{ v \in \mathbb{R}^3 \mid v = \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix},\ z \in \mathbb{R} \right\}$

Systems of Linear Equations

What is a SLE?

$$\begin{cases} 2\boxed{x_1} + 5\boxed{x_2} + 3\boxed{x_3} = \boxed{-3} \\ 4x_1 + 0x_2 + 8x_3 = 0 \\ 1x_1 + 3x_2 + 0x_3 = 2 \end{cases}$$

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Solutions to SLE

1. $$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

2. $$\begin{cases} x + y = 1 \\ 2x + y = 2 \end{cases}$$

3. $$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Solutions to SLE

1. $$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

No solutions.

2. $$\begin{cases} x + y = 1 \\ 2x + y = 2 \end{cases}$$

A single solution: $x = 1,\ y = 0$

3. $$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Infinitely many solutions.

SLE: Matrix Notation

$$\begin{cases} 2x_1 + 5x_2 + 3x_3 = -3 \\ 4x_1 + 0x_2 + 8x_3 = 0 \\ 1x_1 + 3x_2 + 0x_3 = 2 \end{cases}$$

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$$A = \begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$Ax = b$$

SLE: Matrix Notation

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$$Ax = b$$

"Find vector(s) x that are mapped into b by transform A"

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  If we are lucky, $\boldsymbol{b}$ is there, and we have infinitely many solutions.

  If not, there're no solutions.

Solutions to SLE

1. $$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

No solutions.

2. $$\begin{cases} x + y = 1 \\ 2x + y = 2 \end{cases}$$

A single solution: $x = 1,\ y = 0$

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}, \det A \neq 0.$$

3. $$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

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Number of Solutions

- $Ax = b$: "Find vector(s) $\boldsymbol{x}$ that are mapped into $\boldsymbol{b}$ by transform $\boldsymbol{A}$".

- If $\mathbf{det}\,\boldsymbol{A} \neq \mathbf{0}$, there exists A^{-1}: $\quad x = A^{-1}b$.
  So, there is a single solution.

- If $\mathbf{det}\,\boldsymbol{A} = \mathbf{0}$, it's not that simple:

  A maps the original space into a lower dimensional subspace.

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How do we check that?

Number of Solutions

- $Ax = b$ – SLE.
- Consider matrix $(A|b) = \begin{bmatrix} a_{11} & \dots & a_{1n} & b_1 \\ .. & .. & .. & \dots \\ a_{m1} & \dots & a_{mn} & b_n \end{bmatrix}$.

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    - $Ax = b$ has no solutions $\Leftrightarrow rank(A|b) > rank(A)$.

Solutions to SLE

1. $$\begin{cases} x + y = 1 \\ x + y = 2 \end{cases}$$

No solutions.

$$1 = rank(A) < rank(A|b) = 2$$

2. $$\begin{cases} x + y = 1 \\ 2x + y = 2 \end{cases}$$

A single solution: $x = 1,\ y = 0$

$$rank(A) = rank(A|b) = 2$$

3. $$\begin{cases} x + y = 1 \\ 2x + 2y = 2 \end{cases}$$

Infinitely many solutions.

$$rank(A) = rank(A|b) = 1 < 2$$

Gaussian Elimination

Gaussian elimination

- $Ax = b$

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 5 & 6 \\ 2 & 4 & 3 \end{bmatrix}, \qquad b = \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

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- Elementary row operations:

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 3 & 5 & 6 & 7 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 2 & 4 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & -4 & 12 & -8 \\ 0 & -2 & 7 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & -2 & 7 & -2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 9 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -15 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

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Unique solution.

Gaussian Elimination

- $Ax = b$ – SLE.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Gaussian Elimination

- $Ax = b$ – SLE.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 2 & 1 & 5 \end{bmatrix}, \qquad b = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

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girafe ai

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No solutions.

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Gaussian Elimination

- $Ax = b$ – SLE.

$$A = \begin{bmatrix} -3 & -5 & 36 \\ -1 & 0 & 7 \\ 1 & 1 & -10 \end{bmatrix}, \qquad b = \begin{bmatrix} 10 \\ 5 \\ -4 \end{bmatrix}$$

-

girafe ai

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Infinitely many solutions.

Homogeneous SLE

Homogeneous SLE

$$\begin{cases} 2x_1 + 5x_2 + 3x_3 = & \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \\ 4x_1 + 0x_2 + 8x_3 = & \\ 1x_1 + 3x_2 + 0x_3 = & \end{cases}$$

Homogeneous SLE

$$Ax = 0$$

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Solutions = null space of A.

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Solutions = null space of A.

$\operatorname{rank} A = \#variables \rightarrow$ unique solution (0)
$\operatorname{rank} A < \#variables \rightarrow$ infinitely many solutions.

Homogeneous SLE

- $Ax = 0$
- Let V be a set of solutions:

$$\forall v \in V \quad Av = 0$$

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    Indeed, $A(\lambda v) = \lambda A v = 0$
  - $\forall v_1, v_2 \in V \rightarrow (v_1 + v_2) \in V$.
    Indeed, $A(v_1 + v_2) = Av_1 + Av_2 = 0$

Homogeneous SLE

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  - $\forall v_1, v_2 \in V \rightarrow (v_1 + v_2) \in V$.
    Indeed, $A(v_1 + v_2) = Av_1 + Av_2 = 0$

V is a linear subspace!

girafe ai

To sum up

- Matrices as linear transforms
- Examples of common transforms
- Inverse
- Determinant
- Rank
- Solutions to SLE