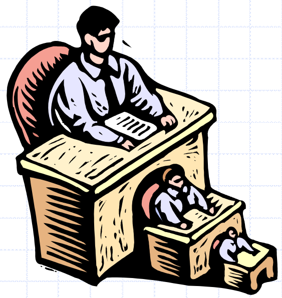
Using Recursion



The Recursion Pattern

- **Recursion**: when a method calls itself
- Classic example--the factorial function:
  - n! = 1· 2· 3· ··· · (n-1)· n
- Recursive definition:

$$f(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot f(n-1) & else \end{cases}$$

- As a C++ method:

```
// recursive factorial function
int  recursiveFactorial(int n) {
   if  (n  ==  0)  return  1;      // basis case
   else return  n  *  recursiveFactorial(n- 1);  // recursive case
}
```

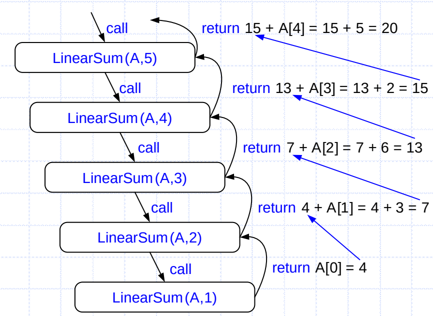

Linear Recursion

- Test for base cases
  - Begin by testing for a set of base cases (there should be at least one).
  - Every possible chain of recursive calls must eventually reach a base case, and the handling of each base case should not use recursion.
- Recur once
  - Perform a single recursive call
  - This step may have a test that decides which of several possible recursive calls to make, but it should ultimately make just one of these calls
  - Define each possible recursive call so that it makes progress towards a base case.

Example of Linear Recursion

Algorithm LinearSum(*A, n*):
Input:
A integer array *A* and an integer *n* = 1, such that *A* has at least *n* elements
Output:
The sum of the first *n* integers in *A*
if *n* = 1 **then**
  return *A*[0]
else
  return LinearSum(*A, n* - 1) + *A*[*n* - 1]

Example recursion trace:



Reversing an Array

Algorithm ReverseArray(*A, i, j*):

Input: An array *A* and nonnegative integer indices *i* and *j*

Output: The reversal of the elements in *A* starting at index *i* and ending at *j*

```
if i < j then
    Swap A[i] and A[j]
    ReverseArray(A, i + 1, j - 1)
return
```


Defining Arguments for Recursion

- In creating recursive methods, it is important to define the methods in ways that facilitate recursion.
- This sometimes requires we define additional paramaters that are passed to the method.
- For example, we defined the array reversal method as ReverseArray(*A, i, j*), not ReverseArray(*A*).

Computing Powers

- The power function, $p(x,n)=x^n$, can be defined recursively:

$$p(x,n) = \begin{cases} 1 & \text{if } n = 0 \\ x \cdot p(x, n-1) & \text{else} \end{cases}$$

- This leads to an power function that runs in O(n) time (for we make n recursive calls).
- We can do better than this, however.

Recursive Squaring

- We can derive a more efficient linearly recursive algorithm by using repeated squaring:

$$p(x,n) = \begin{cases} 1 & \text{if } x = 0 \\ x \cdot p(x,(n-1)/2)^2 & \text{if } x > 0 \text{ is odd} \\ p(x,n/2)^2 & \text{if } x > 0 \text{ is even} \end{cases}$$

- For example,

$2^4 = 2^{(4/2)2} = (2^{4/2})^2 = (2^2)^2 = 4^2 = 16$

$2^5 = 2^{1+(4/2)2} = 2(2^{4/2})^2 = 2(2^2)^2 = 2(4^2) = 32$

$2^6 = 2^{(6/2)2} = (2^{6/2})^2 = (2^3)^2 = 8^2 = 64$

$2^7 = 2^{1+(6/2)2} = 2(2^{6/2})^2 = 2(2^3)^2 = 2(8^2) = 128.$

Recursive Squaring Method

Algorithm Power(x, n):
Input: A number x and integer n = 0
Output: The value x^n
if n = 0 **then**
  return 1
if n is odd **then**
  y = Power(x, $(n - 1)/2$)
  return $x \cdot y \cdot y$
else
  y = Power(x, $n/2$)
  return $y \cdot y$

Analysis

Algorithm Power(x, n):
Input: A number x and integer n = 0
Output: The value x^n
if n = 0 **then**
  return 1
if n is odd **then**
  y = Power(x, $(n - 1)/2$)
  return $x \cdot y \cdot y$
else
  y = Power(x, $n/2$)
  return $y \cdot y$

Each time we make a recursive call we halve the value of n; hence, we make log n recursive calls. That is, this method runs in O(log n) time.

It is important that we use a variable twice here rather than calling the method twice.

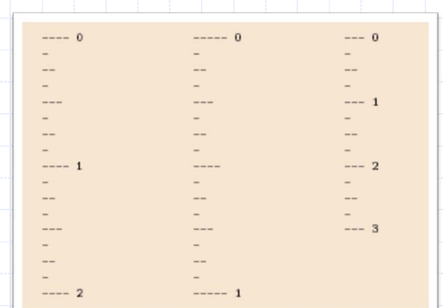
Tail Recursion

- Tail recursion occurs when a linearly recursive method makes its recursive call as its last step.
- The array reversal method is an example.
- Such methods can be easily converted to non-recursive methods (which saves on some resources).
- Example:

Algorithm IterativeReverseArray(A, i, j):
Input: An array A and nonnegative integer indices i and j
Output: The reversal of the elements in A starting at index i and ending at j
while $i < j$ **do**
  Swap $A[i]$ and $A[j]$
  $i = i + 1$
  $j = j - 1$
return

Binary Recursion

- Binary recursion occurs whenever there are **two** recursive calls for each non-base case.
- Example: the DrawTicks method for drawing ticks on an English ruler.



A Binary Recursive Method for Drawing Ticks

```
    // draw a tick with no label
public static void drawOneTick(int tickLength) { drawOneTick(tickLength, - 1); }
    // draw one tick
public static void drawOneTick(int tickLength, int tickLabel) {
  for (int i = 0; i < tickLength; i++)
      System.out.print("-");
  if (tickLabel >= 0) System.out.print(" " + tickLabel);
  System.out.print("\n");
}
public static void drawTicks(int tickLength) { // draw ticks of given length
  if (tickLength > 0) {                        // stop when length drops to 0
      drawTicks(tickLength- 1);      // recursively draw left ticks
      drawOneTick(tickLength);       // draw center tick
      drawTicks(tickLength- 1);      // recursively draw right ticks
  }
}
public static void drawRuler(int nInches, int majorLength) { // draw ruler
  drawOneTick(majorLength, 0);  // draw tick 0 and its label
  for (int i = 1; i <= nInches; i++)        {
      drawTicks(majorLength- 1);  // draw ticks for this inch
      drawOneTick(majorLength, i);            // draw tick i and its label
  }
}
```

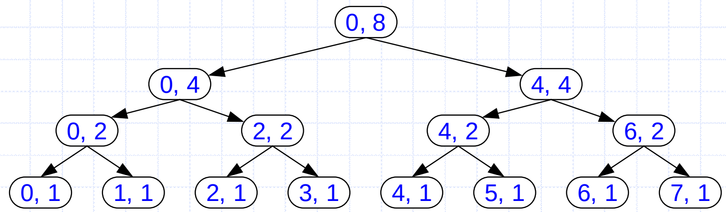
Note the two recursive calls

Another Binary Recusive Method

- Problem: add all the numbers in an integer array A:

  Algorithm BinarySum(*A, i, n*):
  Input: An array *A* and integers *i* and *n*
  Output: The sum of the *n* integers in *A* starting at index *i*
  if $n = 1$ **then**
  return $A[i]$
  return BinarySum$(A, i, n/2)$ + BinarySum$(A, i + n/2, n/2)$

- **Example trace:**



Computing Fibonacci Numbers

- Fibonacci numbers are defined recursively:

  $F_0 = 0$
  $F_1 = 1$
  $F_i = F_{i-1} + F_{i-2}$ for $i > 1$.

- Recursive algorithm (first attempt):

  Algorithm BinaryFib(k):
  Input: Nonnegative integer k
  Output: The kth Fibonacci number F_k
  if $k = 1$ **then**
  return k
  else
  return BinaryFib$(k - 1)$ + BinaryFib$(k - 2)$

Analysis

- Let n_k be the number of recursive calls by BinaryFib(k)
  - $n_0 = 1$
  - $n_1 = 1$
  - $n_2 = n_1 + n_0 + 1 = 1 + 1 + 1 = 3$
  - $n_3 = n_2 + n_1 + 1 = 3 + 1 + 1 = 5$
  - $n_4 = n_3 + n_2 + 1 = 5 + 3 + 1 = 9$
  - $n_5 = n_4 + n_3 + 1 = 9 + 5 + 1 = 15$
  - $n_6 = n_5 + n_4 + 1 = 15 + 9 + 1 = 25$
  - $n_7 = n_6 + n_5 + 1 = 25 + 15 + 1 = 41$
  - $n_8 = n_7 + n_6 + 1 = 41 + 25 + 1 = 67$.
- Note that n_k at least doubles every other time
- That is, $n_k > 2^{k/2}$. It is exponential!

A Better Fibonacci Algorithm

- Use linear recursion instead

Algorithm LinearFibonacci(k):
  Input: A nonnegative integer k
  Output: Pair of Fibonacci numbers (F_k , F_{k-1})
  if $k = 1$ **then**
    return (k, 0)
  else
    (i, j) = LinearFibonacci($k - 1$)
    return (i +j, i)

- LinearFibonacci makes k–1 recursive calls

Multiple Recursion

- Motivating example:
  - summation puzzles
    - *pot + pan = bib*
    - *dog + cat = pig*
    - *boy + girl = baby*
- Multiple recursion:
  - makes potentially many recursive calls
  - not just one or two

Algorithm for Multiple Recursion

Algorithm PuzzleSolve(k,S,U):
Input: Integer k, sequence S, and set U (universe of elements to test)
Output: Enumeration of all k-length extensions to S using elements in U without repetitions
for all e in U **do**
  Remove e from U {e is now being used}
  Add e to the end of S
  if k = 1 **then**
    Test whether S is a configuration that solves the puzzle
    if S solves the puzzle **then**
      return "Solution found: " S
  else
    PuzzleSolve(k - 1, S,U)
  Add e back to U {e is now unused}
  Remove e from the end of S

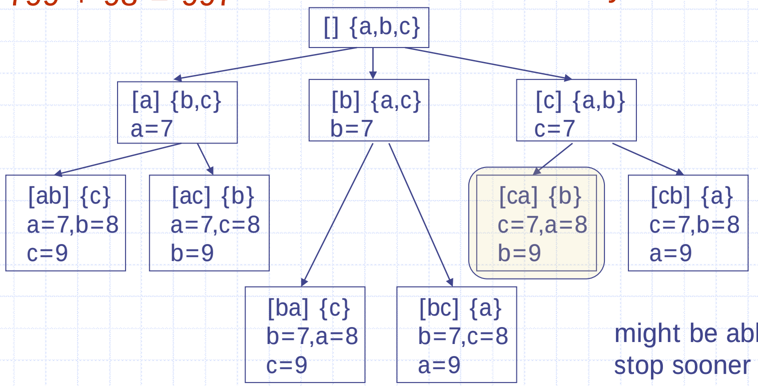
Slide by Matt Stallmann included with permission.

Example

cbb + ba = abc

799 + 98 = 997

a,b,c stand for 7,8,9; not necessarily in that order



might be able to stop sooner

Visualizing PuzzleSolve